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ABSTRACT

This course is intended for students who have a thorough knowledge of college preparatory mathematics, including algebra, axiomatic geometry, trigonometry, and analytic geometry. This text, Part II, contains material designed to follow Part I. Chapters included in this text are: (6) Derivatives of Exponential and Related Functions; (7) Area and the Integral; (8) Differentiation Theory and Technique; and (9) Integration Theory and Technique. Appendices include: (3) Mathematical Induction; (4) Further Techniques of Integration; (5) The Integral for Monotone Functions; (6) Inequalities and Limits; (7) Continuity Theorems; (8) More about Integrals; and (9) Logarithm and Exponential Functions as Solutions to Differential Equations.

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SCHOOL
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CALCULUS OF ELEMENTARY FUNCTIONS

Part II

Student Text

(Revised Edition)

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CALCULUS OF ELEMENTARY FUNCTIONS

Part II

Student Text

(Revised Edition)

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FOREWORD TO PART II

In Chapters 1, 3, and 5 of Part I we studied polynomial, circular, exponential, logarithmic, and power functions. As we saw in Chapters 2 and 4, many properties of the graph of a function can be obtained from the knowledge of the derivative of the function, since the value of the derivative can be interpreted as the slope of the tangent line at a point. For polynomial and circular functions we were able to find derivatives, which were then used to determine rise and fall, convexity and concavity, velocity, and acceleration.

In Chapter 5, we shall do the same for exponential and related functions. These methods are extended in Chapter 8 to functions which are sums, products, composites, powers, reciprocals, quotients or inverses of known functions. We study functions which are various algebraic combinations of previously studied functions and we develop the general theory and techniques of differential calculus which can be used to handle all but the most pathological of functions.

In Chapter 7 of Part II we begin the study of area under the graph of a function. The area concept, at first glance, seems to be unrelated to that of the slope of a tangent line to a graph. The discovery of the relationship between these two concepts was one of the great breakthroughs in mathematics, first noted by Barrow (1630-1677). Barrow was able to show that the area bounded by the graph of f, the x-axis and vertical lines at a and b is given by $F(b) - F(a)$, where F is a function whose derivative is f. This result is a simple geometric interpretation of a profound relationship known as the Fundamental Theorem of Calculus.

Chapter 8 is primarily a discussion of methods of differentiating algebraic combinations of functions. Where appropriate, integration concepts (that is, the area concepts of Chapter 7) are also discussed, since they provide a further geometric interpretation for describing the behavior of functions.

Integration concepts and techniques are explored further in Chapter 9, which contains an important method for finding antiderivatives, an interpretation of the Fundamental Theorem in terms of average value, volumes of solids of revolution, numerical methods, and a discussion of remainder estimates for Taylor approximations.

The appendices are intended to fill logical gaps in the intuitive development of the text and to extend the material of the text, concluding with Appendix 9 in which logarithmic and exponential functions are viewed as solutions of simple differential equations. It is shown how the expression of the logarithm as an integral can be used to obtain the properties of logarithmic and exponential functions.

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Chapter 6

DERIVATIVES OF EXPONENTIAL AND RELATED FUNCTIONS

The derivative of a polynomial function is again a polynomial function. Furthermore, the derivative of a circular function is again a circular function. This kind of repetitive property appears in a very strong form for exponential functions, for the slope of the tangent line at a point on the graph of an exponential function is proportional to the ordinate of the point. The constant of proportionality is the slope of the tangent line at the point where the graph crosses the y-axis. The number e is defined as the base for which the constant of proportionality is 1, from which it follows that the derivative of $x \to e^x$ is the same function $x \to e^x$. These results are established in the first two sections of this chapter as consequences of the laws of exponents and the assumption that $x \to 2^x$ has a derivative at $x = 0$.

Logarithm functions were defined in Chapter 5 as inverses of exponential functions. This inverse relation enables us to differentiate a logarithm function by a folding process (Section 6-5). Using the fact that a power function can be expressed in terms of exponential and logarithm functions we are then able to find a formula for the derivative of a power function (Section 6-6). The concept of polynomial approximation, first discussed for circular functions in Chapter 4, is then extended to exponential, logarithm, and root functions (Section 6-7).

6-1. The Tangent Line to the Graph of $x \to a^x$ at $(0, a^0)$

Now we wish to find the slope of the tangent line to the graph of $x \to a^x$ at some arbitrary point on this curve. Our procedure for polynomials and for the circular functions was to first find the equation or slope of the tangent line at the point where the curve crosses the vertical axis and then translate to obtain the corresponding results elsewhere. This procedure will also be followed here. In our previous discussions we found the tangent as the line of best fit; we then showed that the slope of the tangent at a point is obtainable as the limit of slopes of lines connecting the point under consideration to nearby points. We shall follow this latter limit process here.

To be concrete we first consider the problem of finding the slope of the tangent to the graph of $x \to 2^x$ at the point where $x = 0$. If $|h|$ is small the line connecting $A(0,2^0)$ to $B(h,2^h)$ will approximate the slope of the tangent line at $A(0,2^0)$. (See Figure 6-1a)

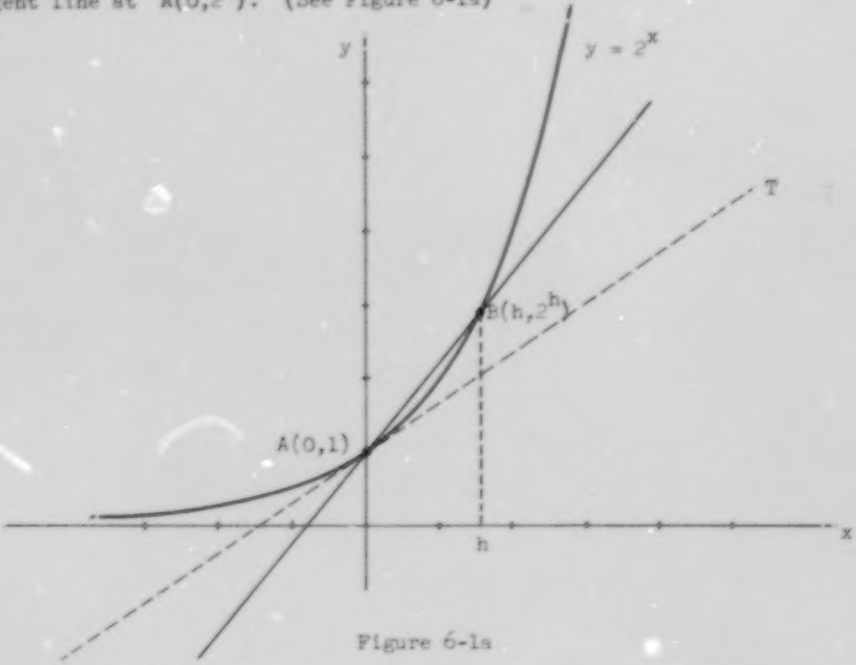


Figure 6-1a

If B is close to A, the slope of $\overleftrightarrow{AB}$ approximates the slope of T, the tangent at (0,1).

The line $\overleftrightarrow{AB}$ has slope

$$\frac{2^h - 1}{h}. \tag{1}$$

We want to find the number this ratio approximates when $|h|$ is small.

Table 6-1

Values of $\frac{2^h - 1}{h}$ for small h (correct to 3 places)

h	2^h	$2^h - 1$	$\frac{2^h - 1}{h}$
.10	1.07177	.07177	.718
.05	1.03526	.03526	.705
.01	1.0069556	.0069556	.696
.005	1.0034717	.0034717	.694
.001	1.0006934	.0006934	.693
-.05	.96594	-.03406	.681
-.01	.9930925	-.0069075	.691
-.005	.9965402	-.0034598	.692
-.001	.9993071	-.0006929	.693

Table 6-1 indicates some of the values of (1) for small h. It appears from the table that

if $|h|$ is small, then

$$\frac{2^h - 1}{h} \approx k, \tag{2}$$

where, to three places, k is 0.693.

While this approximation to k is correct, we need more than a table of values, no matter how complete, to be certain. Unfortunately, we have no simple algebraic device for determining the limit of this ratio as h approaches zero. We are assuming that the graph of $x \to 2^x$ has a tangent at $(0,1)$ and that the slope of this tangent is approximated by $\frac{2^h - 1}{h}$. We shall assume that (2) is true and concentrate on the consequences of this assumption.

If (2) is true we have that the slope of the tangent line to the graph of $x \to 2^x$ at $(0,2^0)$ is k. At $(0,2^0) = (0,1)$ the equation of the tangent is

$$y = 1 + kx. \tag{3}$$

For $|x|$ close to zero we have

$$2^x \approx 1 + kx. \tag{4}$$

Now consider the function

$$x \to a^x \quad \text{where} \quad a > 0, \quad a \neq 1.$$

In Chapter 5 we saw that we can express a as a power of 2. If $a = 2^{\alpha}$ we can write

(5) $$a^x = 2^{\alpha x}.$$

If we assume that $|x|$ is so small that $|\alpha x|$ is small, then we can replace x by αx in (4) and use (5) to obtain

(6) $$a^x \approx 1 + k(\alpha x)$$

In other words, the line with equation

$$y = 1 + (k\alpha)x$$

is the tangent to the graph of $x \to a^x$ at the point $(0,1)$. The coefficient of x is the slope of this line, so the slope of the tangent to $x \to a^x$ at $x = 0$ is $k\alpha$.

For example, since $4 = 2^2$, the tangent line to the graph of $x \to 4^x$ at $x = 0$ has the equation

$$y = 1 + 2kx.$$

Also, since $\frac{1}{\sqrt{2}} = 2^{-1/2}$, the tangent to the graph of $x \to \left(\frac{1}{\sqrt{2}}\right)^x$ at $x = 0$, has the equation

$$y = 1 - \frac{k}{2}x.$$

The respective slopes of these lines are $2k$ and $-\frac{k}{2}$.

In our discussion of the circular functions we saw that we could select our scale (using radians, rather than degree measure) so that the slope of the tangent to $y = \sin x$ at $x = 0$ turned out to be 1. Similarly here we shall obtain considerable simplification in our formulas if we choose α in (6) so that $k\alpha = 1$. With $k\alpha = 1$ we have $\alpha = \frac{1}{k}$. Thus if $a = 2^{1/k}$ then our result (6) gives

$$a^x \approx 1 + x, \quad \text{if} \quad |x| \quad \text{is small;}$$

that is, the slope of the tangent to $x \to a^x$ at $x = 0$ is 1.

The number $2^{1/k}$ is so important that a special letter is assigned to it, namely e. We can approximate e by

$$e = 2^{1/k}, \quad \text{where } k \approx .693.$$

This gives the approximation

$$e^x \approx 1 + x \quad \text{if } |x| \text{ is small.} \tag{7}$$

If $|h|$ is small then the slope of the tangent to the graph of $x \to e^x$ at $(0,1)$ is

$$\frac{e^h - 1}{h} \approx 1. \tag{8}$$

The use of e in this sense may be traced to the Swiss mathematician Leonard Euler (1707 - 1783). Most of Euler's mathematical life was spent in St. Petersburg, Russia. His work is still being collected and at present numbers more than 80 volumes. The number e ranks in importance with the number π and is, curiously enough, closely related to π.

If we use 0.693 to approximate k we obtain

$$\frac{1}{k} \approx \frac{1}{0.693} \approx 1.443$$

so that

$$\begin{aligned} e = 2^{1/k} &\approx 2^{1.443} = 2(2^{0.4})(2^{0.04})(2^{0.003}) \\ &\approx 2(1.320)(1.028)(1.002) \\ &\approx 2.72. \end{aligned}$$

Closer approximations to k will obviously improve this approximation. In recent years, high speed computers have been used to obtain the decimal expansion of e correct to 2500 places. For the record, we note that the first 15 places are given by

$$e = 2.71828 \quad 18284 \quad 59045.... \tag{9}$$

For our purposes either 2.72 or 2.718 will be good enough.

The number e has been shown to be irrational, just as is $\sqrt{2}$. In fact, a much stronger result has been established, namely it has been shown that e is not the root of a polynomial equation with rational coefficients. The same is true for π. (The number $\sqrt{2}$ is such a root; e.g., it is a root of $x^2 - 2 = 0$.)

There is an important method for approximating e, given as follows

$$e \approx \left(1 + \frac{1}{n}\right)^n \quad \text{for } n \text{ large.} \tag{10}$$

This is a consequence of (*) for if n is large, $\frac{1}{n}$ is small so that

$$e^{1/n} \approx 1 + \frac{1}{n} .$$

We give two equivalent definitions:

$$\boxed{\begin{aligned} e &= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \\ e &= \lim_{h \to 0} (1 + h)^{1/h} \end{aligned}}$$

The function $x \to e^x$ is called the exponential function in distinction to all other exponential functions. The exponential function is often denoted by exp.

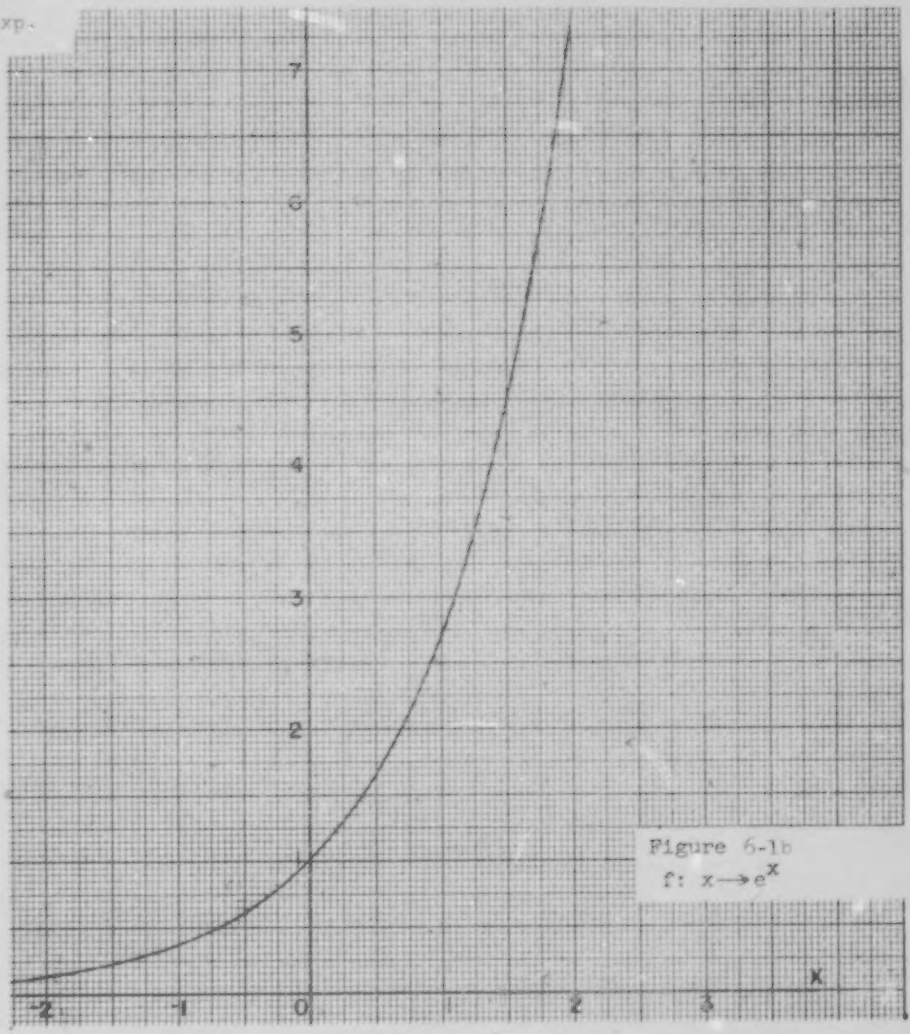


Figure 6-1b
f: $x \longrightarrow e^x$

Exercises 6-1

1. Given the function

$$f : x \to a^x \quad \text{for} \quad a = 8,\ \frac{1}{8},\ \sqrt{8},\ \frac{1}{\sqrt{8}}.$$

   (a) Find the slopes of the tangent at $(0,1)$ to the graph of the function for each value of $\underline{a}$

      (i) in terms of k, where $k \approx \frac{2^h - 1}{h}$ for small $|h|$;

      (ii) as an approximate value, using $k \approx 0.693$.

   (b) Find the equations of the tangents, for which the slopes were obtained in part (a).

   (c) On one set of axes for each value of $\underline{a}$ given above, sketch the graph of

      (i) the function;

      (ii) the tangent obtained in part (b).

2. Given $(1.8)^5$

   (a) Using the table for values of 2^h,

      (i) express $(1.8)^5$ as a power of 2;

      (ii) approximate the value of $(1.8)^5$ from 2a(i).

   (b) Using the table for e^x and e^{-x}

      (i) express $(1.8)^5$ as a power of e;

      (ii) approximate the value of $(1.8)^5$ from 2b(i).

3. Follow the instructions of Number 2 for $(0.9)^5$.

4. Follow the instructions of Number 2 for $(1.02)^8$.

5. Obtain bounds for $(1.01)^{100}$, using the table for Values of 2^h as follows:

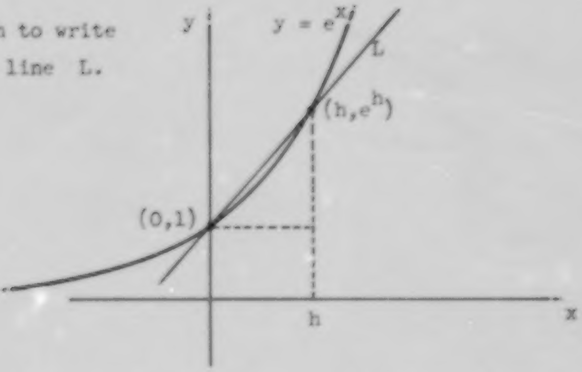
   (a) Write $(1.01)^{100}$ as an inequality

$$(2^{\alpha_1})^{100} < (1.01)^{100} < (2^{\alpha_2})^{100}$$

   (b) Evaluate $2^{100\alpha_1}$ and $2^{100\alpha_2}$, thereby obtaining upper and lower bounds for $(1.01)^{100}$.

6. Obtain bounds for $(0.5)^{-12}$, using the table for e^x and e^{-x}, and following a procedure similar to that of Number 5.

7. (a) Consult the sketch to write the slope m_0 of line L.



(b) Write an expression for e if $m_0 = 1$ and $h \neq 0$.

(c) Use your expression from part (b) and binomial expansion to give an approximation for e to one decimal place if $h = .01$.

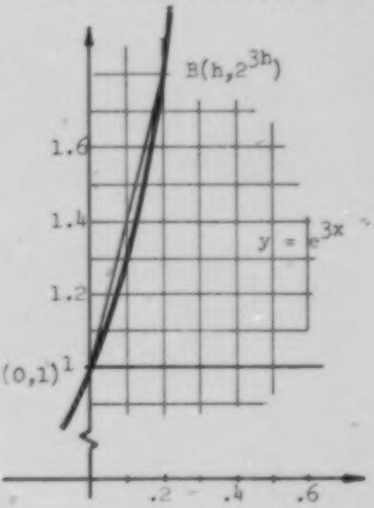
(d) Improve upon the result of part (b) to show that e may be defined as the limit of $(1 + \frac{1}{n})^n$ as we let n grow large without bound.

8. Which is larger 1000^{1001} or 1001^{1000}?

9. Show that at $x = 0$ the slope of the tangent to the graph of the function

$$f : x \rightarrow e^{3x}$$

is close to 3 when $|h|$ is close to zero, by completing the following tables.

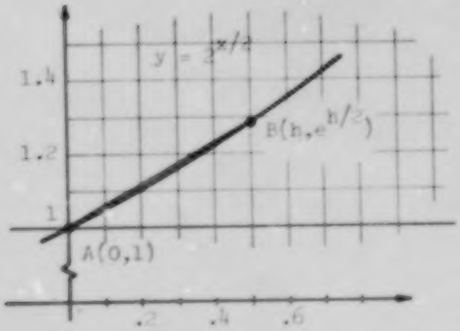


h	3h	e^{3h}	$e^{3h} - 1$	$\frac{e^{3h} - 1}{h}$
.20				
.15				
.10				
.05				
.01				
.006				

10. Show that at $x = 0$ the slope of the tangent to the graph of the function

$$f : x \rightarrow e^{x/2}$$

is close to $\frac{1}{2}$ when $|h|$ is close to zero, by completing these tables.

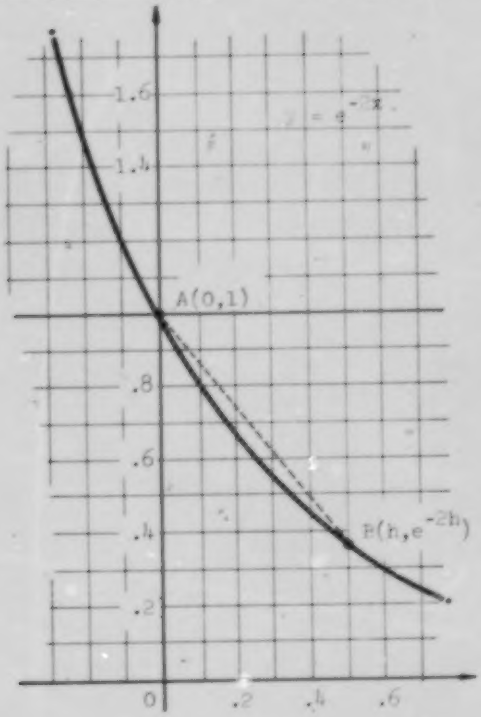


h	$\frac{h}{2}$	$e^{h/2}$	$e^{h/2} - 1$	$\frac{e^{h/2} - 1}{h}$
.50				
.40				
.30				
.20				
.10				
.06				
.02				

11. Show that at $x = 0$ the slope of the tangent to the graph of the function

$$f : x \rightarrow e^{-2x}$$

is close to -2 when $|h|$ is close to zero, by completing the following tables.



h	$2h$	e^{-2h}	$e^{-2h} - 1$	$\frac{e^{-2h} - 1}{h}$
.20				
.15				
.10				
.05				
.02				
.01				
.005				

12. In one of the problems of Number 1 we found the slope of the equation of the tangent at the point $(0,1)$ to the graph of the function

$$f : x \to 8^x.$$

In terms of an inequality, approximate this slope to four significant figures after filling in the following tables.

(In the first table, h approaches zero through positive values from the right, and, in the second table, h approaches zero through negative values from the left.)

h	$3h$	2^{3h}	$2^{3h} - 1$	$\frac{2^{3h} - 1}{h}$
.20				
.15				
.10				
.05				
.01				
$.00\dot{6}$				
$.000\dot{6}$				
-.20				
-.15				
-.10				
-.05				
-.01				
$-.00\dot{6}$				
$-.000\dot{6}$				

6-2. The Tangent at an Arbitrary Point

In the previous section we obtained the result

$$a^x \approx 1 + (k\alpha)x, \quad \text{for } |x| \text{ small,} \tag{1}$$

where $a = 2^\alpha$ and k is the limit of $\frac{2^h - 1}{h}$ as h approaches 0.

With $a = e$ we have the simpler result

$$e^x \approx 1 + x.$$

We shall now show that the tangent line to the graph of

$$f : x \to e^x$$

at $P(a, e^a)$ is

$$y = e^a + e^a(x - a) \tag{2}$$

so that the slope of the graph at P is e^a, the same as the ordinate e^a. (See Figure 6-2a.)

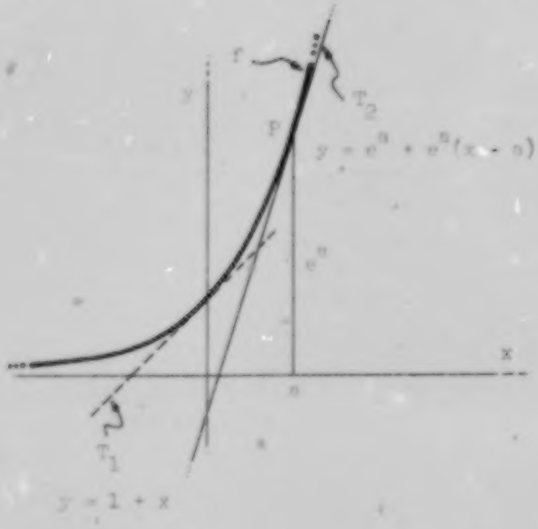


Figure 6-2a

Graph of $y = e^x$ with tangents T_1 and T_2

To prove this result we put

$$x = a + (x - a) \quad \text{in} \quad e^x.$$

Then

$$e^x = e^{a+(x-a)} \qquad (3)$$
$$= e^a \cdot e^{x-a}$$

If x is close to a, $x - a$ is close to 0 and hence,

$$e^{x-a} \approx 1 + (x - a).$$

Substituting this result in (3) gives

$$e^x \approx e^a + e^a(x - a). \qquad (4)$$

The tangent to the graph of $x \to e^x$ at (a,e^a) has the equation $y = e^a + e^a(x - a)$.

(5) $\boxed{\text{At the point } (a,e^a) \text{ the slope of the tangent to the graph of } x \to e^x \text{ is } e^a.}$

As in our previous discussion, the resulting slope function is called the derivative. That is, the derivative of $x \to e^x$ is the function whose value at x is the slope of the tangent line at (x,e^x). We restate (5) using derivative terminology.

(6) $\boxed{\text{If } f : x \to e^x, \text{ then the derivative } f' \text{ is given by } f' : x \to e^x.}$

In particular, $f : x \to e^x$ is a solution to the **differential equation**

$$f' = f. \qquad (7)$$

Example 6-2a. Find the equation of the tangent to the graph of $f : x \to e^x$ at the point $(3,e^3)$.

For $f : x \to e^x$ we have the derivative

$$f' : x \to e^x$$

so that $f'(3) = e^3$. The tangent to the graph of f at $(3,e^3)$ with slope e^3 has the equation

$$y = e^3 + e^3(x - 3).$$

Exercises 6-2

1. Use the data in the table of e^x and e^{-x} to find the slope of the tangent to the graph of $f : x \to e^x$ at the following points.

   (a) $(-1, e^{-1})$

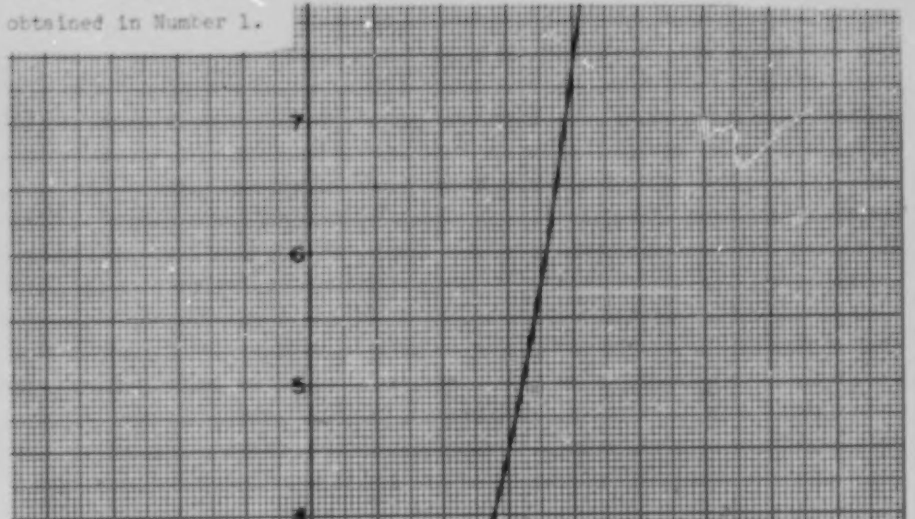
   (b) $(0.5, e^{0.5})$

   (c) $(0.7, e^{0.7})$

   (d) $(0, 1)$

   (e) $(1.5, e^{1.5})$

2. Use the graph of $f : x \to e^x$ below to estimate the slope of the tangent at the points given in Number 1. Compare your results with those obtained in Number 1.



3. Write an equation of the tangent to the graph of f at each point (x,e^x) given in Number 3.

4. (a) Through the point $(3,4)$ draw a line L_1 with slope $m = \frac{2}{5}$.

   (b) Draw a line L_2 which is symmetric to L_1 with respect to the y-axis.

   (c) What point on L_2 corresponds to the point $(3,4)$ on L_1?

   (d) What is the slope of L_2?

   (e) Consider the general case: line L_1 drawn through point (r,s) with slope $= m$, and line L_2 symmetric to L_1 with respect to the y-axis. What point on L_2 corresponds to point (r,s) on L_1? What is the slope of L_2?

5. (a) Plot the points (x,e^x) for which $x = -2.0, -1.8, \ldots, 0.2, 0.4, \ldots, 1.6$.

   (b) Through each of these points draw the graph of a line having slope $m = e^x$.

   (c) Show that these lines suggest the shape of the graph of $f : x \to e^x$.

6. (a) For each point plotted in Number 5(a) locate the corresponding point which is symmetric with respect to the y-axis; then through these points draw lines symmetric to those of Number 4(b) with respect to the y-axis.

   (b) Show that each point located in Number 6(a) lies on the graph of $g : x \to e^{-x}$.

   (c) Compare the slopes of the lines drawn in Number 6(a) with those of Number 5(b).

7. (a) On one set of coordinate axes draw the graphs of $f : x \to e^x$ and $g : x \to e^{-x}$.

   (b) Compare the slopes of the graphs drawn in (a) at $x = 0, +1, -1$.

   (c) Compare the slope of the graph of g at $x = h$ with $g(h)$.

6-3. Applications of Exponential Functions.

Exponential functions arise in practice in the study of growth or decay. We discuss compound interest in this section and give a few other applications in the exercises.

Compound interest. Suppose that P dollars is invested at an annual rate of interest of r per cent or $\frac{r}{100}$, and at the end of each year interest is compounded, or added to the principal. After t years the total amount A_t on hand is given by

$$A_t = P(1 + \frac{r}{100})^t.$$

However, the interest may be compounded semiannually, quarterly, or n times a year. If interest is added to the principal n times per year, the rate of interest is $\frac{r}{100n}$ per period, and the number of periods in t years is nt. Hence, the amount A_{nt} after nt periods (that is, after t years) is

$$A_{nt} = P(1 + \frac{r}{100n})^{nt}. \tag{1}$$

The more often you compound interest, the more complicated the calculation becomes. On the other hand, if we let n in (1) get larger indefinitely, we approach the theoretical situation in which interest is compounded continuously; we shall see that the result obtained will enable us to find easily a very satisfactory approximation for the amount of money on hand at the end of a reasonable period of time.

To study this idea, let $\frac{r}{100n} = h$ so that $n = \frac{r}{100h}$. Then (1) becomes

$$\begin{aligned} A_{nt} &= P(1 + h)^{rt/100h} \\ &= P[(1 + h)^{1/h}]^{rt/100}. \end{aligned} \tag{2}$$

For large n, the value of h approaches zero and the right side of (2) approximates

$$A = Pe^{rt/100},$$

the theoretical amount that would be obtained if interest were compounded continuously at r per cent. Thus

$$A = Pe^{rt/100}. \tag{3}$$

Example 6-3a. If \$100 is invested at 4 percent for 10 years, compare the amount after 10 years when interest is compounded continuously with the amount after 10 years if interest is compounded only annually.

We have $P = 100$, $r = 4$, and $t = 10$ (years). If interest is compounded continuously, (3) gives

$$A = 100e^{0.4},$$

which is approximately 149.

To compute interest compounded annually we substitute the above values of P, r, and t in (1). This gives

$$A_{10} = 100(1.04)^{10}.$$

We may use a table of common logarithms to estimate A_{10}; thus

$$A_{10} \approx 100(1.48) = 148.$$

The results, \$149 and \$148, differ by a surprisingly small amount.

Exercises 6-3

1. When his son Jack was born, Mr. Toffey invested \$1000 for Jack's college education. Interest is compounded continuously at a rate of 3 per cent. How much money will Mr. Toffey have for Jack's education on Jack's eighteenth birthday?

2. Using $2 = e^{0.693}$, find how many years it takes to double a sum of money invested at 3 per cent compounded continuously.

3. Jack Toffey (of No. 1) earns a scholarship and elects to wait and to withdraw his father's investment when it has doubled. How old will Jack be when he withdraws the \$2000?

4. Determine how many years it will take to double a sum of money invested at

   (a) 6 per cent compounded continuously;

   (b) n per cent compounded continuously.

5. The quantity $(1 + \frac{1}{n})^n$ can be interpreted as the value at the end of one year of a deposit of one dollar left to acquire interest at an annual interest rate of 100% compounded n times a year. If the interest is compounded continuously, that is, if the interest is calculated as the limit in which the number n of interest periods approaches infinity, the value of the principal at the end of one year will be e dollars, \$2.72.

   (a) A California savings and loan association offers an interest rate of 4.85% compounded continuously. What is the equivalent annual interest rate for money left on deposit one year?

   (b) How long does it take for an amount of money at the same interest rate (4.85% compounded continuously) to double itself?

6. At h kilometers above sea level, the pressure in millimeters of mercury is given by the formula

$$P = P_0 e^{-0.11445h}$$

where P_0 is the pressure at sea level. If $P_0 = 760$, at what height is the pressure 180 millimeters of mercury?

7. A law frequently applied to the healing of wounds is expressed by the formula

$$Q = Q_0 e^{-nr},$$

where Q_0 is the original area of the wound, Q is the area that remains unhealed after n days, and r is the so-called rate of healing. If $r = 0.12$, find the time required for a wound to be half-healed.

8. If light of intensity I_0 falls perpendicularly on a block of glass, its intensity I at a depth of x feet is

$$I = I_0 e^{-kx}.$$

If one third of the light is absorbed by 5 feet of glass, what is the intensity 10 feet below the surface? At what depth is the intensity $\frac{1}{2} I_0$?

6-4. The Derivative of a Logarithmic Function

The graph of the logarithmic function

$$x \to \log_a x, \quad a > 0, \quad a \neq 1,$$

can be obtained by folding the graph of

$$x \to a^x$$

over the line $y = x$. Just as in the previous section we can use this folding process to find the derivative of $x \to \log_a x$. We discuss first the important case when $a = e$.

Suppose (c,d) is a point on the graph of

$$f : x \to \log_e x$$

so that $\log_e c = d$. Hence,

$$(1) \qquad c = e^d$$

so that (d,c) lies on the graph of

$$g : x \to e^x.$$

The tangent line L_1 to graph of g at the point (d,c) has slope $g'(d)$, where g' is the derivative of g. Since

$$g' : x \to e^x$$

we have $g'(d) = e^d$, the slope of the tangent L_1 to the graph of g at (d,c). The process of folding over the line given by $y = x$ carries L_1 into the tangent line L to the graph of the logarithmic function f at the point (c,d). (See Figure 6-4a.)

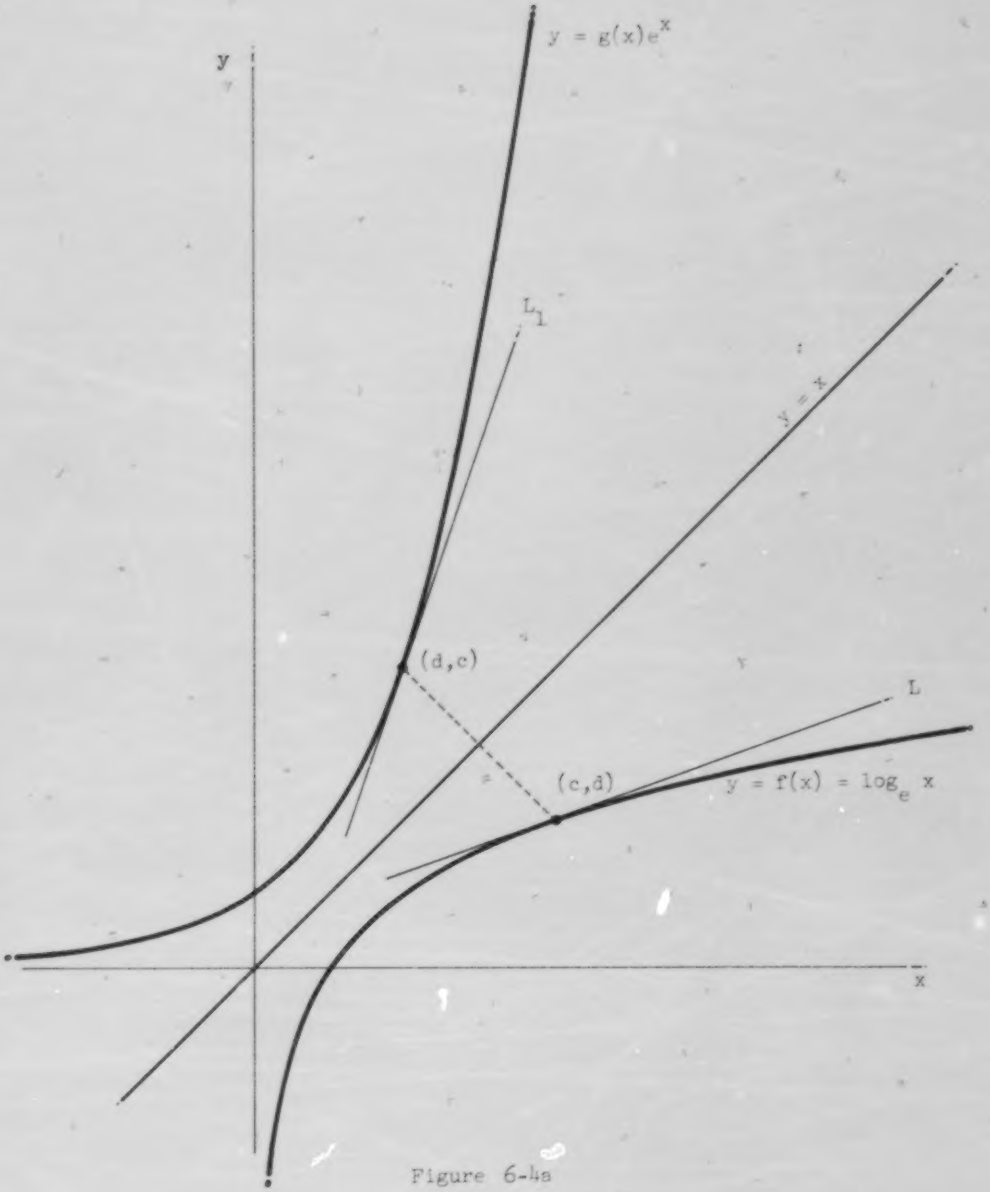

y = g(x)e^x
y
L_1
y = x
(d,c)
L
(c,d)
y = f(x) = $\log_e$ x
x

Figure 6-4a

Tangent line L_1 folds over the line given by $y = x$ onto tangent line L.

The slope of L is the reciprocal of the slope of L_1 so that

$$\text{the slope of } L = \frac{1}{e^d} .$$

The slope of L is the value of the derivative of $f : x \to \log_e x$ at the point where $x = c$. Thus

$$f'(c) = \frac{1}{e^d}\ .$$

To express this in terms of c, we use (1) to replace e^d by c, obtaining

$$f'(c) = \frac{1}{c}\ .$$

In general we can say, for $x > 0$,

(2) $$\boxed{\begin{array}{l} \text{if} \quad f : x \to \log_e x \\ \text{then} \quad f' : x \to \frac{1}{x} \end{array}}$$

We can rewrite (2) as

(3) $$\boxed{D(\log_e x) = \frac{1}{x}}$$

The tangent line to the graph of f at the point (c,d) has slope $f'(c) = \frac{1}{c}$ and passes through $(c,d) = (c, \log_e c)$. Hence the equation of the tangent line is

$$y = \log_e c + \frac{1}{c}(x - c).$$

If x is close to c the tangent line serves to approximate the curve and we have

(4) $$\log_e x \approx \log_e c + \frac{1}{c}(x - c).$$

The derivative of the general logarithm function $\log_a$ can be obtained by a process similar to that which we used to derive (3). It is also a simple consequence of a relation derived earlier, namely

$$\log_a x = \frac{\log_e x}{\log_e a}\ , \quad \text{if} \quad a > 0, \quad a \neq 1.$$

In fact

$$D \log_a x = \frac{1}{\log_e a} \cdot \frac{1}{x}$$

Example 6-4a. Find the equation of the tangent line to the graph of $x \to \log_e x$ at the point where $x = e$.

Knowing that $\log_e e = 1$, we see that $(e,1)$ lies on the graph of $x \to \log_e x$. Since $D \log_e x = \frac{1}{x}$, the slope at $(e,1)$ is $\frac{1}{e}$. Hence the tangent at $(e,1)$ has the equation

$$y = 1 + \frac{1}{e}(x - e).$$

The function $x \to \log_e x$ is referred to in most advanced works as the logarithmic function and denoted simply by log without subscript. Common logarithms (logarithms with base 10) are still useful for hand computation but with the advent of machine computation they have lost much of their once great importance. The logarithms used in analysis are almost invariably logarithms with base e and are referred to as "natural" logarithms.

In most elementary texts $\log x$ means $\log_{10} x$ and $\ln x$ means $\log_e x$, in most professional literature $\log x$ means $\log_e x$; in this text we shall try to avoid ambiguity by specifying the base of a logarithm unless the context makes the base perfectly clear.

John Napier (1550-1617) is justly regarded as the inventor of the logarithmic function. Although the basic idea was definitely "in the air" of his times, he was the first to publish a table of a logarithmic function (1614) and his ideas about logarithms were more insightful and efficient for the construction of tables than those of his contemporaries. Napierian logarithms, usually thought to be logarithms to the base e, are in fact given by

$$\text{Napierian } \log x = 10^7 \log_{1/e} \left(\frac{x}{10^7}\right).$$

Henry Briggs (1561-1631) was largely responsible for the introduction of logarithms with base 10 for the purposes of computation.

A table of natural logarithms (logarithms to the base e) is contained in the accompanying Booklet of Tables (Table 6). We can use this table to compute logarithms not contained in it, if we apply the properties of logarithm functions.

Example 6-4b. Find $\log_e 1.44$. Since $1.44 = (1.2)^2$

$$\begin{aligned}\log_e 1.44 = \log_e (1.2)^2 &= 2 \log_e 1.2\\ &\approx 2(0.1823)\\ &\approx 0.3646.\end{aligned}$$

We can also perform computations using these properties and the Table.

Example 6-4c. Compute $\sqrt{3}$ approximately

$$\begin{aligned}\log_e \sqrt{3} = \log_e 3^{1/2} &= \frac{1}{2} \log_e 3\\ &\approx \frac{1}{2} \cdot 1.0986\\ &\approx .5493.\end{aligned}$$

Since $\log_e 1.7 \approx 0.5306$ and $\log_e 1.8 \approx 0.5878$. $\sqrt{3}$ is between 1.7 and 1.8. Interpolating, $\sqrt{3} \approx 1.73$.

Exercises 6-4

1. Using the table of natural logarithms find the approximate numerical value for each of the following:

   (a) $\log_e(1.96)$ [Hint: $1.96 = (1.4)^2$]

   (b) $\log_e(2.03)$ [Hint: $2.03 = (2.9)(.7)$]

   (c) $\log_e(0.52)$ in two ways:

   (i) $\log_e(0.52) = \log_e(\frac{3.9}{7.5})$

   (ii) $\log_e(0.52) = \log_e(\frac{5.2}{10})$

   (d) $\log_e(0.052)$.

   (e) $\log_e(\frac{750,000}{39,000,000})$

2. Using the tables for natural logarithms find the approximate numerical value for each of the following:

   (a) $\sqrt{2}$ (c) $(9.1)^{2/3}$

   (b) $\sqrt[3]{71}$ (d) $(100)^{1/2}$

3. For some x close to c, we have by (5) $\log_e x$ approximated by $\log_e c + \frac{1}{c}(x - c)$. Using only this formula and the table value, $\log_e 2 \approx .6931$, find the following logarithms:

   (a) $\log_e(2.01)$ (c) $\log_e(2.03)$

   (b) $\log_e(1.96)$ (d) $\log_e(1.94)$

4. Using the results of Number 3, find an approximate value for each of the following:

   (a) $(2.01)^{5/3}$ (c) $(2.03)^{0.6}$

   (b) $\sqrt[6]{1.96}$ (d) $(1.94)^{1.1}$

5. (a) What is the x-intercept of the following?

   (i) $x \to \log_e 3x$ (iv) $x \to \log_e \frac{x}{2}$

   (ii) $x \to \log_e 2x$ (v) $x \to \log_e \frac{x}{3}$

   (iii) $x \to \log_e x$ (vi) $x \to \log_e \frac{x}{4}$

(b) Given: $x \to \log_e kx$, $k(\text{constant}) > 1$.

(i) The x-intercept must be in what interval?

(ii) As k gets very large, what does the x-intercept approach?

(c) Given: $x \to \log_e \frac{x}{k}$, $k(\text{constant}) > 1$.

(i) The x-intercept must be in what interval?

(ii) As k gets very large, what does the x-intercept approach?

6. (a) For a given abscissa, what is the vertical distance between each of the following?

(i) $x \to \log_e 2x$ and $x \to \log_e x$

(ii) $x \to \log_e 3x$ and $x \to \log_e 2x$

(iii) $x \to \log_e 4x$ and $x \to \log_e 3x$

(iv) $x \to \log_e (k+1)x$ and $x \to \log_e kx$ $(k > 1)$

(b) In Number 6(a)(iv) above, as k gets very large, what effect does this have on the vertical distance?

7. (a) For a given abscissa, what is the vertical distance between each of the following?

(i) $x \to \log_e x$ and $x \to \log_e \frac{x}{2}$

(ii) $x \to \log_e \frac{x}{2}$ and $x \to \log_e \frac{x}{3}$

(iii) $x \to \log_e \frac{x}{3}$ and $x \to \log_e \frac{x}{4}$

(iv) $x \to \log_e \frac{x}{k}$ and $x \to \log_e \frac{x}{k+1}$ $(k > 1)$

(b) In Number 7(a)(iv) above, as k gets very large, what effect does this have on the vertical distance?

8. (a) Find the derivative of the following functions by using (4) and the property, $\log ab = \log a + \log b$. [Hint: Remember that the derivative of a constant is zero.]

(i) $x \to \log_e 2x$ (iv) $x \to \log_e \frac{x}{3}$

(ii) $x \to \log_e \frac{x}{2}$ (v) $x \to \log_e kx$, $k > 0$

(iii) $x \to \log_e 3x$ (vi) $x \to \log_e \frac{x}{k}$, $k > 0$

(b) Find the slope of each of the curves represented in part (a) of Number 8 at the point where $x = e$.

(c) Find the coordinates of the point on each curve above where $x = e$.

(d) Find the equation of the tangent line to each of the curves at $x = e$.

(e) (i) What are the y-intercepts of each tangent?

(ii) Show that the y-intercepts of the tangents to $x \rightarrow \log_e kx$ and $x \rightarrow \log_e \frac{x}{k}$ $(k > 1)$ are symmetric with respect to the origin.

(f) Sketch carefully the following on one graph using the same set of axes, for the region: $0 < x < 3.5$, $-3 < y < 2$:

$f : x \rightarrow \log_e x$, and its tangent at $x = e$;

$f : x \rightarrow \log_e 2x$, and its tangent at $x = e$; and

$f : x \rightarrow \log_e \frac{x}{2}$, and its tangent at $x = e$.

Indicate, (where possible,)

x- and y-intercepts of logarithm curves
x- and y-intercepts of tangent lines
parallelism of tangents
vertical distance between tangents
vertical distance between logarithm curves.

9. Using the law of logarithms: $\log a^b = b \log a$.

(a) Find the derivative of the following

(i) $x \rightarrow \log_e x^2$

(ii) $x \rightarrow \log_e x^3$

(iii) $x \rightarrow \log_e \sqrt{x}$

(iv) $x \rightarrow \log_e \sqrt[3]{x}$

(b) Show that

(i) $D \log_e x^n = \frac{n}{x}$

(ii) $D \log_e \sqrt[n]{x} = \frac{1}{nx}$

(iii) $D \log_e (cx + d)^n = \frac{nc}{cx + d}$

(iv) $D \log_e \sqrt[n]{cx + d} = \frac{c}{n(cx + d)}$

10. Using the results of Number 9 above find the derivative of the following functions. [Hint: When formulas do not seem to apply, remember the laws of logarithms: $\log ab = \log a + \log b$, $\log \frac{a}{b} = \log a - \log b$, $\log a^b = b \log a$, $\log \frac{1}{a} = -\log a$.]

(a) $x \to \log_e (5x + 1)^3$

(b) $x \to \log_e (4x^2 \sqrt{x})$

(c) $x \to \log_e x(1 - 2x)$

(d) $x \to \log_e x^2(3x - 1)$

(e) $x \to \log_e [\log_e e^x]$

(f) $x \to \log_e (\sin \frac{\pi}{2})$

(g) $x \to \log_e \frac{2x - 1}{2x + 1}$

(h) $x \to \log_e \sqrt{\frac{1 + x}{1 - x}}$

11. Find the equation of the only tangent to the graph of $y = \log_e x$ that passes through the origin. Compare your equation with the result of Example 6-4a.

6-5. Taylor Approximations to the Function $x \to e^x$

The derivative of $x \to e^x$ is $x \to e^x$. Thus the second and higher derivatives of $x \to e^x$ are also $x \to e^x$. In other words, if $f(x) = e^x$, then

$$e^x = f'(x) = f''(x) = \dots = f^{(n)}(x) = \dots \tag{1}$$

Just as we did for the sine function we now seek to find polynomials with the same derivatives as $x \to e^x$. More specifically, we wish to find a polynomial p such that

(2)

(a) the degree of p does not exceed n

(b) $p(0) = 1 = e^0$

(c) the values of the first n derivatives of p and $x \to e^x$ are the same for $x = 0$.

For example, consider the case for which $n = 3$. We put $p(x) = a + bx + cx^2 + dx^3$. We have

$$p'(x) = b + 2cx + 3dx^2,$$
$$p''(x) = 2c + 6dx,$$
$$p'''(x) = 6d;$$

so that

$$p(0) = a, \quad p'(0) = b, \; p''(0) = 2c, \quad p'''(0) = 6d.$$

Suppose $f : x \to e^x$, so that

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 1.$$

Hence, if p satisfies (2) then

$$1 = a, \quad 1 = b, \quad 1 = 2c, \quad 1 = 6d;$$

so that p is necessarily given by

$$p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

In general, we have

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \tag{3}$$

as the unique polynomial which satisfies (2). These polynomials are called the Taylor* approximations to e^x.

---

*Brook Taylor - English 1685-1731.

We ask the same question that we did for the sine function. How good are the Taylor approximations? Figure 6-5a indicates the graph of $x \to e^x$ and the third degree Taylor approximations $x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Just as with the sine function, as the degree increases the Taylor approximations to $x \to e^x$ become better in the sense that subsequent approximations give better approximations near zero and give good approximations further away from zero.

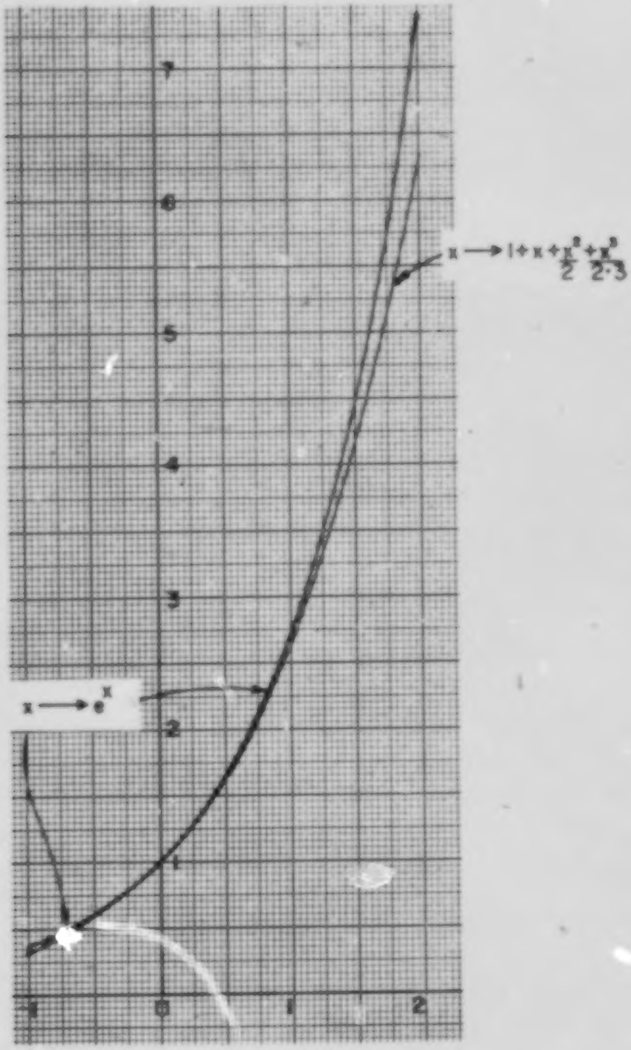


Figure 6-5a

The graph of $x \to e^x$ and its Third Degree Polynomial Approximation.

In Chapter 9 we shall use area principles to establish the result:

$$(4) \qquad e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n,$$

where the remainder term R_n satisfies the inequality

$$(5) \qquad R_n \leq \frac{e^M x^{n+1}}{(n+1)!} \quad \text{if} \quad 0 \leq x \leq M.$$

Thus, for example, if $0 \leq x \leq 1$, then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + R_3,$$

where
$$R_3 \leq \frac{e\,x^4}{4!};$$

and
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + R_4,$$

where
$$R_4 \leq \frac{e\,x^5}{5!}.$$

Formulas (4) and (5) are useful in constructing exponential tables. We observe that it is necessary to find e^x only for $0 \leq x \leq 1$. Larger powers can be calculated from knowledge of these. For example, if we know $e^{0.13}$, then we can find $e^{2.13}$ by using the relation

$$e^{2.13} = e^2 \cdot e^{0.13}.$$

Negative powers can be obtained by taking reciprocals. Thus $e^{-1.3} = \frac{1}{e^{1.3}}$.

Suppose we wish to construct tables of e^x for $0 \leq x \leq 1$, correct to two decimal places. We first choose n large enough so that the error term R_n cannot affect the first two places. We observe that on $[0,1]$, $e^1 < 3$, so that formula (5) gives

$$R_n \leq \frac{e\,x^{n+1}}{(n+1)!} < \frac{3}{(n+1)!}.$$

We can therefore estimate correctly to two decimal places if we choose n so large that $\frac{3}{(n+1)!} < 0.005$.

Rewriting we get $\frac{3}{(n+1)!} < \frac{5}{1000}$ or $600 < (n+1)!$. Since $6! = 720$, we can choose $n = 5$ and then know that using the formula

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

will give answers correct to two decimal places for $0 \le x \le 1$.

Example 6-5a. Find $e^{0.1}$ correct to three decimal places.

We first estimate (5) with $M = 0.1$. We know that $e^{.1} < e^1 < 3$, so we need only choose n so large that on $[0,1]$

$$R_n \le \frac{e^{0.1}\ x^{n+1}}{(n+1)!} < \frac{3}{(n+1)!}\ 10^{-(n+1)} < .0005.$$

We have

$$\frac{3}{4!}\ 10^{-4} = 0.125 \times 10^{-4} < .0005.$$

Thus we know that, correct to three decimal places,

$$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!}$$

$$= 1 + \frac{1}{10} + \frac{1}{200} + \frac{1}{6000} \approx 1.105.$$

We can also use (4) and (5) to obtain limits as x approaches zero of various expressions involving e^x. The next example illustrates this method.

Example 6-5b. Find the limit of

$$\frac{(1 - e^x)(1 - \cos x)}{x^3} \quad \text{as } x \text{ approaches zero.}$$

We shall do this first in a rough way.

Since $e^x \approx 1 + x$ and $\cos x \approx 1 - \frac{x^2}{2!}$,

$$(1 - e^x)(1 - \cos x) \approx -x\left(\frac{x^2}{2}\right) = -\frac{x^3}{2}.$$

Hence,
$$\frac{(1 - e^x)(1 - \cos x)}{x^3} \approx -\frac{1}{2}$$

and the required limit is $-\frac{1}{2}$.

More precisely we can take account of the errors made in using the approximations to e^x and $\cos x$ if we use the remainders R_1 and R_2 in

$$e^x = 1 + x + R_1$$

$$\cos x = 1 - \frac{x^2}{2!} + R_2 .$$

Then

$$(1 - e^x)(1 - \cos x) = (-x - R_1)(\frac{x^2}{2} - R_2)$$

$$= - \frac{x^3}{2!} + x R_2 - \frac{x^2}{2} R_1 + R_1R_2$$

$$= x^3\left[-\frac{1}{2} + \frac{R_2}{x^2} - \frac{R_1}{2x} + \frac{R_1R_2}{x^3}\right].$$

Since $0 < R_1 < \frac{ex^2}{2!}$ for x on $[0,1]$ and

$$0 < R_2 < \frac{x^4}{4!}$$

then $\frac{R_2}{x^2}$, $\frac{R_1}{2x}$ and $\frac{R_1R_2}{x^3}$ approach 0 and $\lim_{x \to 0} \frac{(1 - e^x)(1 - \cos x)}{x^3} = -\frac{1}{2}$.

The result (4) can also be used to show that if x is large enough, $e^x > x^k$ no matter how large the exponent k may be (k a positive integer).

From (4), for any $x > 0$,

$$e^x > \frac{x^n}{n!} .$$

Let $n = k + 1$. Then

$$e^x > \frac{x^{k+1}}{n!}$$

and

$$\frac{e^x}{x^k} > \frac{x}{n!} .$$

This means that when $x > n!$

$$\frac{e^x}{x^k} > 1,$$

that is,

$$e^x > x^k.$$

Exercises 6-5

1. Write the first four terms of a polynomial approximation for each of the following.

(a) e^x (d) $\cos x$

(b) $-e^x$ (e) $-\cos x$

(c) $1 - e^x$ (f) $1 - \cos x$

For Numbers 2 through 5 consider the graph of each function. Write the polynomial function, which serves as the best

(a) linear
(b) quadratic
(c) cubic

approximation to the graph of the function near the y-axis.

2. $f : x \rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

3. $g : x \rightarrow y = \sin x$

4. $F : x \rightarrow y = \cos x$

5. $G : x \rightarrow y = e^x$

6. Do you suppose that there are polynomial functions that can serve to approximate the graph of

$$f : x \rightarrow y = \log_e x$$

at the y-axis? Explain.

7. Compute $e^{0.01}$ correct to five decimal places. Obtain the value of each term to six places, continuing until you reach terms which have only zeros in the first six places, add, and round off to five places. How many terms did you need to use? Note that even though the remaining terms are individually less than 0.000001, they might accumulate to give a very large sum; in this particular case, they do not.

8. Obtain an approximation to e by computing successively

$$e^{0.2} = (e^{0.1})^2, \quad e^{0.4}, \quad e^{0.8}, \quad e = (e^{0.2})(e^{0.8}).$$

Use the estimate $e^{0.1} \approx 1.105$ of Example 6-5a.

9. Suppose

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

(a) Show that $p'_n(x) = p_n(x) - \frac{x^n}{n!}$.

(b) Show that $p'_n(x) < p_n(x)$ if $x > 0$.

(c) Deduce from (b) that $p_n(x) < e^x$ if $x > 0$.
(Hint: Observe that at $x = 0$ both functions start the same. Then determine what affect the slopes have upon the graphs when $x > 0$.)

10. Suppose $c > 1$ and

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{cx^3}{3!}, \quad p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

(a) If $x > 0$ show that $g(x) > p_3(x)$.

(b) Show that if $0 < x < 3\left(\frac{c - 1}{c}\right)$, then $g'(x) > g(x)$ for $x > 0$.

(c) (i) If $x = 2$ then $c < 2 < \frac{3(c - 1)}{c}$. What is the smallest integer which satisfies this conclusion?

(ii) If $f : x \to e^x$ show that $g(2) > f(2)$.

(d) By an argument involving the comparison of the slopes of f and g show that $g(x) > f(x)$ for $0 < x < \frac{3(c - 1)}{c}$.

11. Let

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$g_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n - 1)!} + \frac{cx^n}{n!}, \quad \text{where } c > 1.$$

Show that

$$p_n(x) < e^x < g_n(x) \quad \text{for} \quad 0 < x < \frac{n(c - 1)}{c}.$$

(Hint: See Nos. 9, 10, above.)

12. Using the functions $p_n(x)$ and $q_n(x)$ as well as the results of Number 11, deduce that

$$0 < e^x - p_n(x) < \frac{(c - 1)x^n}{n!} \quad \text{if} \quad 0 < x < \frac{n(c - 1)}{c}, \; c > 1.$$

13. What degree must the Taylor approximation be to give e^x for $|x| \leq 2$ correct to two decimal places? three decimal places? (Use the estimate $e^2 < 9$.)

14. What degree must the Taylor approximation be to give e^x for $|x| \leq 0.5$, correct to four decimal places? (Use the estimate $e^{0.5} < 2$.)

15. Find $e^{0.001}$ correct to five decimal places. Do the same for $e^{-0.001}$.

16. (a) Replace x by cx to obtain approximations to e^{cx} of degree ≤ 5.

    (b) Find a polynomial approximation to e^{x^2} of degree ≤ 8.

17. Find the limit of each of the following expressions as x approaches 0.

    (a) $\dfrac{(1 - e^{-x^2})\sin x}{x^3}$

    (b) $\dfrac{e^x - \cos x}{x}$

    (c) $\dfrac{\cos x^2 - e^{x^4}}{\sin x^3}$

18. Find $\lim\limits_{x \to 1} \dfrac{e - e^x}{x - 1}$.

6-6. The Power Formula

The result for the derivative of $x \to e^x$ enables us to find the derivatives of the so-called power functions

$$f : x \to x^r,$$

where r is any real number, rational or irrational. We know from Chapter 2 that if $r = n$, a positive integer, then

$$f' : x \to nx^{n-1}.$$

It is remarkable that f' is given by the corresponding formula for any real number r, so that

$$f' : x \to rx^{r-1}.$$

We shall prove this important result:

$$(1) \qquad \boxed{\begin{array}{ll} \text{If} & f : x \to x^2 \\ \text{then} & f' : x \to rx^{r-1}. \end{array}}$$

We start with the remark that for any positive number z

$$e^{\log_e z} = z.$$

If, in particular, $z = x^r$

$$e^{\log_e x^r} = x^r.$$

Since

$$\log_e x^r = r \log_e x$$

$$(2) \qquad x^r = e^{r \log_e x}.$$

For x near some number, say b, we have the best linear approximation,

$$\log_e x \approx \log_e b + \frac{1}{b}(x - b).$$

Multiplying by r, we get

$$r \log_e x \approx r \log_e b + \frac{r}{b}(x - b).$$

From (2), we have

$$x^r \approx e^{r \log_e b + \frac{r}{b}(x - b)}.$$

Thus, $$x^r \approx e^{r \log_e b} \cdot e^{\frac{r}{b}(x - b)},$$

according to the law of exponents. Since $e^{r \log_e b} = e^{\log_e b^r} = b^r$, we can

write $$x^r \approx b^r e^{\frac{r}{b}(x - b)}.$$

Now for $\frac{r}{b}(x - b)$ near 0

$$e^{\frac{r}{b}(x - b)} \approx 1 + \frac{r}{b}(x - b)$$

and therefore x^r is approximately

$$b^r[1 + \frac{r}{b}(x - b)] = b^r + \frac{b^r r}{b}(x - b)$$
$$= b^r + r\, b^{r-1}(x - b).$$

Thus, $y = b^r + rb^{r-1}(x - b)$ is the equation of the tangent line to the graph of $y = x^r$ at (b, b^r). The slope of the tangent is rb^{r-1}. This is the value of the derivative at b.

We have, therefore, established (1) for the case $x > 0$; that is, we have shown that

(1) if $f : x \to x^r$, then $f' : x \to rx^{r-1}$.

This is the case which is most important in practice. The formula (1) is also correct when $x = 0$ if $r > 1$.

For $x < 0$, f is undefined unless r is rational with $r = \frac{m}{n}$, m and n non-negative integers, n odd. In this case, (1) holds but we shall not prove this statement here.

Example 6-6a. Find the derivative of $f : x \to \frac{1}{x}$, defined for $x \neq 0$. We can write

$$f(x) = \frac{1}{x} = x^{-1}$$

and use (1) to obtain the derivative

$$f': x \to (-1)x^{-2} = -\frac{1}{x^2},$$

valid for any $x \neq 0$.

Note that the derivative of $x \to \frac{1}{x}$ is always negative; that is, any tangent to its graph has negative slope. Intuitively it is clear from this that $x \to \frac{1}{x}$ is a decreasing function for all $x \neq 0$.

The derivative in this case can also be obtained by using simple algebra. The line connecting $(x, \frac{1}{x})$ to $(x + h, \frac{1}{x + h})$ has slope

$$\begin{aligned}\frac{\frac{1}{x + h} - \frac{1}{x}}{x + h - x} &= \frac{1}{h}[\frac{1}{x + h} - \frac{1}{x}] \\ &= \frac{1}{h}[\frac{x - (x + h)}{(x + h)x}] \\ &= \frac{1}{h}[\frac{-h}{(x + h)x}] \\ &= \frac{-1}{(x + h)x}.\end{aligned}$$

This difference quotient approaches $-\frac{1}{x^2}$ as h approaches 0.

Example 6-6b. Find the equation of the tangent to the graph of $f : x \to x^{3/2}$ at the point where $x = 4$.

Formula (1) gives

$$f'(x) = \frac{3}{2}x^{\frac{3}{2} - 1} = \frac{3}{2}x^{1/2}.$$

If $x = 4$, then

$$f(4) = 4^{3/2} = (\sqrt{4})^3 = 8$$

and

$$f'(4) = \frac{3}{2}(4)^{1/2} = \frac{3}{2}\sqrt{4} = 3.$$

The equation of the tangent to the graph of f at $(4,8)$ is

$$y = 8 + 3(x - 4).$$

Example 6-6c. Find the derivative of $x \to x^{\sqrt{2}}$.

Formula (1) gives the derivative

$$x \to \sqrt{2}\, x^{\sqrt{2}-1}.$$

Since $\sqrt{2} - 1 > 0$, this is valid for $x \geq 0$.

Example 6-6d. Find the derivative of $x \to \sqrt{x}$. Since $\sqrt{x} = x^{1/2}$, we have from (1)

$$D\, x^{1/2} = \frac{1}{2}\, x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}},$$

valid when $x > 0$.

This result may also be obtained from the definition of the derivative

$$D\sqrt{x} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

if we transform the difference quotient by multiplying by $\dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$.

Then $\dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \dfrac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$. The limit is $\dfrac{1}{\sqrt{x} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$.

We can generalize the Power Formula to enable us to

(a) multiply by any constant k,

(b) change x to $x - a$, where a is a constant.

We shall show that

$$Dkx^r = krx^{r-1} \tag{3}$$

and

$$Dk(x-a)^r = kr(x-a)^{r-1}. \tag{4}$$

Hereafter, we shall refer to (4) as the Power Formula. Previously, we have used this term for the special case, $k = 1$ and $a = 0$.

To establish (3), we let $f(x) = kg(x)$. Then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{kg(x+h) - kg(x)}{h} \\ &= k\,\frac{g(x+h) - g(x)}{h}. \end{aligned}$$

Allowing h to approach 0, we have the result

$$(5) \qquad Df(x) = kDg(x).$$

If in particular $g(x) = x^r$, we have established

$$(3) \qquad Dkx^r = krx^{r-1}.$$

To establish (4), we let

$$h : x \to k(x - a)^r$$

and

$$f : x \to kx^r.$$

The graph of h is the result of translating the graph of f by the amount a. (See Figure 6-6a)

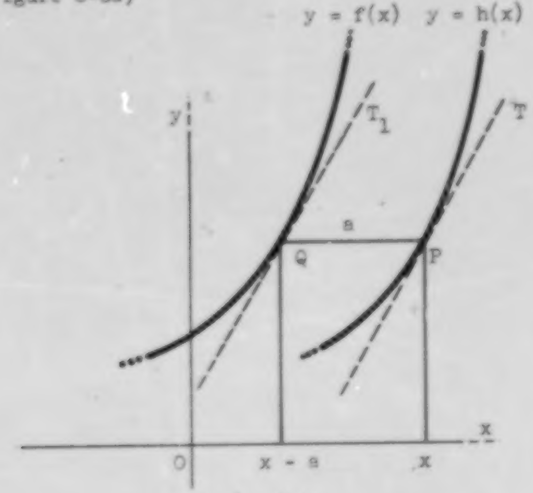


Figure 6-6a

The result of translating the graph of f.

At the point $P(x,h(x))$ the tangent T has the slope $h'(x)$. At the point $Q(x - a, f(x - a))$, the tangent T_1 has the slope $f'(x - a)$. Since T_1 and T are parallel, the two slopes are equal. Therefore,

$$(6) \qquad h'(x) = f'(x - a).$$

Since

$$f'(x) = krx^{r-1}$$

$$f'(x - a) = kr(x - a)^{r-1}.$$

Hence, finally,

$$h'(x) = kr(x - a)^{r-1},$$

that is,

$$Dk(x - a)^r = kr(x - a)^{r-1}. \tag{4}$$

Example 6-6e. Find the derivatives of $(x - 1)^{-1}$, $(x - 1)^{-2}$, $(x - 1)^{-3}$.

Since

$$\begin{aligned} Dx^{-1} &= (-1)x^{-2}, \\ Dx^{-2} &= (-2)x^{-3}, \qquad\qquad x \neq 0 \\ Dx^{-3} &= (-3)x^{-4}, \end{aligned}$$

then

$$\begin{aligned} D(x - 1)^{-1} &= (-1)(x - 1)^{-2} \\ D(x - 1)^{-2} &= (-2)(x - 1)^{-3} \qquad\qquad x \neq 1 \\ D(x - 1)^{-3} &= (-3)(x - 1)^{-4}. \end{aligned}$$

Exercises 6-6

1. Find the derivatives of the following functions.

   (a) $x \to 2x^{3/2}$

   (b) $x \to \frac{6}{\sqrt{x}}$

   (c) $x \to \frac{5}{2} x^{2/5}$

   (d) $x \to (\frac{x}{10})^{1/10}$

   (e) $x \to \sqrt{2x}$

   (f) $x \to \frac{4}{\sqrt[3]{8x^2}}$

   (g) $x \to \frac{1}{2} \sqrt{\frac{1}{2x}}$

   (h) $x \to 20(\frac{3x}{\pi})^{(.7)}$

   (i) $x \to 2 \cdot \frac{\sqrt[3]{x}}{\sqrt{2x}}$ [Hint: Simplify first!]

   (j) $x \to \frac{4}{3} \cdot \frac{1}{x}$

2. For what values of x are the above functions (No. 1) defined?

3. For what values of x are the derivatives of the above functions (No. 1) defined?

4. Find the slope of the curves (described by the functions of No. 1) at $x = 1$, and at $x = 2$.

5. Which of the functions in Number 1 are defined at $x = 0$?

6. Which of the derivatives found in Number 1 are defined at $x = 0$?

7. Find the derivative of the following functions:

   (a) $x \to \sqrt{x + 1}$

   (b) $x \to \sqrt[3]{x - 4}$

   (c) $x \to \frac{1}{(x + 2)^3}$

   (d) $x \to \sqrt{\frac{4}{x}}$

   (e) $x \to \sqrt{2x + 3} = \sqrt{2} \sqrt{x + \frac{3}{2}}$

   (f) $x \to \frac{\sqrt{x - 1}}{\sqrt{x^2 - 1}}$

   (g) $x \to \frac{b}{\sqrt{cx + d}}$ b, c, d positive constants

8. For what values of x are the above (No. 7) functions defined?

9. For what values of x are the derivatives of the above functions (No. 7) defined?

10. Given: $f : x \to 2\sqrt{1 - x}$

    (a) Find f': Then find $f'(-8)$, $f'(-3)$, $f'(2)$.

    (b) f is defined for what interval of x.

    (c) f' is defined for what interval of x.

    (d) For what interval of x is f increasing? decreasing?

    (e) Find the equation of the tangent to the curve at $x = 0$.

    (f) Sketch the curve, and the tangent at $x = 0$.

11. Given: $f : x \to \sqrt[3]{x^2}$

    (a) Find f'.

    (b) When is f decreasing? increasing?

    (c) Find the equation of the tangent at $x = 1$.

    (d) A tangent to the curve is parallel to $x + y = 2$. Find the equation of this tangent line.

    (e) Is there a tangent line at $x = 0$? If so, what is its equation?

    (f) Sketch the graph of the curve, and the tangent line at $x = 1$.

12. Given: $f : x \to x - \frac{1}{x}$.

    (a) For what values of x is f increasing? decreasing?

    (b) What happens to the curve when $|x|$ gets larger?

    (c) Find the equation of the tangent(s) parallel to the line $5x - y = 0$.

    (d) If the curve is tangent to $y = mx + b$ ($m \neq b$, constants) at some point on the curve, find the values which m can assume.

    (e) Sketch the graph.

13. (a) Find the first three derivatives p', p'', p''' of the polynomial function

    $$p : x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^9}{9!} + \frac{x^{10}}{10!}\ .$$

    (b) Evaluate $p(0)$, $p'(0)$, $p''(0)$, and $p'''(0)$.

    (c) Guess the derivative of

    $$f : x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

6-7. Approximations to Logarithmic and Root Functions

As we try to use polynomial functions to approximate logarithmic and power functions a new situation arises: the functions we try to approximate or their derivatives may not be defined at $x = 0$. Our usual procedure of first considering the graph of a function at the y-axis may be inappropriate. We can avoid this problem by considering approximations to such a function at other points, or we can find the appropriate Taylor approximations to a translated function.

Approximations to $\log_e (1 + x)$

At $x = 0$ the function $x \to \log_e x$ is not defined, so we shall consider the translated function

$$f : x \to \log_e (1 + x).$$

This process gives the subsequent derivatives:

$$f^{(4)} : x \to -2 \times 3(1 + x)^{-4} = -\frac{3!}{(1 + x)^4}$$

$$f^{(5)} : x \to 2 \times 3 \times 4(1 + x)^{-5} = \frac{4!}{(1 + x)^5}$$

$$\vdots$$

$$f^{(k)} : x \to (-1)^{k-1}(k - 1)!(1 + x)^{-k} = \frac{(-1)^{k-1}(k - 1)!}{(1 + x)^k},$$

where k is an integer greater than or equal to 1. We let $x = 0$ in each of these to obtain the values

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1,$$

$$f'''(0) = 2!, \quad f^{(4)}(0) = -3!, \quad f^{(5)}(0) = 4!,$$

and in general

$$f^{(k)}(0) = (-1)^{k-1}(k - 1)!, \quad k \geq 1. \tag{1}$$

Suppose n is a fixed positive integer and that

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n.$$

As in Section 6-6 the values $p_n(0)$, $p_n'(0)$, ..., are given by

$$p_n(0) = a_0, \quad p_n'(0) = a_1, \quad p_n''(0) = 2a_2, \quad p_n'''(0) = 6a_3$$

and in general

(2) $$p_n^{(k)}(0) = k!\ a_k .$$

If p_n is to be the Taylor approximation to $f : x \to \log_e(1 + x)$ we must have

$$p_n(0) = f(0), \quad p_n'(0) = f'(0), \quad p_n''(0) = f''(0), \ \ldots$$

so that

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = -\frac{1}{4}, \ \ldots$$

In general we equate (1) with (2) to obtain for $k \geq 1$

$$k!\ a_k = (-1)^{k-1}(k - 1)!$$

so that

(3) $$a_k = \frac{(-1)^{k-1}}{k}, \quad \text{if} \quad k \geq 1.$$

Therefore,

(4) $$\boxed{p_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n} .}$$

We can use (4) to give the Taylor polynomials for

$$x \to \log_e (1 + x)$$

to any prescribed accuracy. In Section 9-5 we shall show that for $x \geq 0$ the error R_n satisfies the inequality

(5) $$|R_n| \leq \frac{x^{n+1}}{n + 1} .$$

If n is large and $0 \leq x \leq 1$ $\frac{x^{n+1}}{n+1}$ is very small. Thus, we can expect the error estimate to be small in the interval $0 \leq x \leq 1$ if we use high degree polynomial approximation. For $x > 1$, powers of x become very large so that the error estimate gives a large error. (Of course, we cannot then conclude that R_n is large, only that the estimate of R_n is large. It is, however, true that R_n is very large when x is larger than 1 and n is large.)

Example 6-7a. Use Taylor approximations of fifth degree to estimate $\log_e 2$.

With $n = 5$, the Taylor approximation for $x \to \log_e(1 + x)$ is

$$\log_e(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

and for $x \geq 0$ the error R_n satisfies

$$|R_n| \leq \frac{x^6}{6} .$$

We let $x = 1$ to obtain

$$\log_e 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \approx 0.783$$

with error at most $\frac{1^6}{6} = \frac{1}{6}$. This is not very good. To guarantee accuracy to within 0.005 we could use (5) to show that we must choose n to be at least 199.

Example 6-7b. Use Taylor approximations of third degree to estimate $\log_e 1.1$.

For $x = 0.1$ and $n = 3$

$$\log_e 1.1 \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} \approx .09533$$

with error at most $\frac{(0.1)^4}{4} = .000025$, so that the estimate is correct to 4 places.

Approximations to $\sqrt{1 + x}$

At $x = 0$ the derivative of the function $x \to \sqrt{x}$ is not defined, so we consider the translated function $f : x \to \sqrt{1 + x}$. The power formula (1) gives the successive derivatives

$$\begin{aligned}
f' &: x \to \tfrac{1}{2}(1 + x)^{-1/2} \\
f'' &: x \to (\tfrac{1}{2})(-\tfrac{1}{2})(1 + x)^{-3/2} \\
f''' &: x \to \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})(1 + x)^{-5/2} \\
f^{(4)} &: x \to \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})(1 + x)^{-7/2}.
\end{aligned}$$

To suggest a pattern we rewrite these in the form

$$f'' : x \to \frac{1}{2}(\frac{1}{2} - 1)(1 + x)^{\frac{1}{2} - 2}$$

$$f''' : x \to \frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(1 + x)^{\frac{1}{2} - 3}$$

$$f^{(4)}: x \to \frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3)(1 + x)^{\frac{1}{2} - 4},$$

so that, in general, for $k \geq 1$,

$$f^{(k)} : x \to \frac{1}{2}(\frac{1}{2} - 1) \ldots [\frac{1}{2} - (k - 1)](1 + x)^{\frac{1}{2} - k}$$

These give the values

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}$$

$$f''(0) = \frac{1}{2}(\frac{1}{2} - 1)$$

$$f'''(0) = \frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)$$

$$f^{(4)}(0) = \frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3)$$

and, in general, for $k \geq 1$,

(6) $$f^{(k)}(0) = \frac{1}{2}(\frac{1}{2} - 1) \ldots (\frac{1}{2} - k + 1).$$

Suppose

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

so that, as we found in (2):

$$p^{(k)}(0) = k!\, a_k, \quad k = 0, 1, 2, \ldots .$$

Equating $p(0) = f(0)$, $p'(0) = f'(0)$, ..., $p^{(n)}(0) = f^{(n)}(0)$ gives

$$a_0 = 1$$

$$a_1 = \frac{1}{2}$$

$$a_2 = \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} = -\frac{1}{8}$$

$$a_3 = \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!} = \frac{1}{16}$$

$$a_4 = \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3)}{4!} = \frac{-5}{128}$$

and, in general, for $k = 1, 2, \ldots, n$

$$a_k = \frac{\frac{1}{2}(\frac{1}{2} - 1) \ldots (\frac{1}{2} - k + 1)}{k!} . \tag{7}$$

These give the coefficients of the Taylor approximations to $x \to \sqrt{1 + x}$. For example, if $n = 4$ we have

$$p(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + R_4$$

as the polynomial which agrees with $\sqrt{1 + x}$ at $x = 0$ and whose first four derivatives agree with the first four derivatives of $x \to \sqrt{1 + x}$ at $x = 0$.

As before, for each positive integer n, we let $p(x)$ be the corresponding Taylor approximation to $x \to \sqrt{1 + x}$. The remainder R_n is then given by

$$R_n = \sqrt{1 + x} - p(x).$$

Estimates for R_n are usually somewhat complicated. We content ourselves with stating one result:

$$|R_n| \leq |a_{n+1}|\, x^{n+1}, \quad \text{if} \quad 0 \leq x \leq 1, \tag{8}$$

where

$$a_{n+1} = \frac{\frac{1}{2}(\frac{1}{2} - 1) \ldots (\frac{1}{2} - n)}{(n + 1)!} .$$

Example 6-7c. Use the Taylor approximation with $n = 4$, to estimate $\sqrt{\frac{3}{2}}$.

We have

$$\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + R_4,$$

where

$$|R_4| \leq |a_5|x^5, \quad 0 \leq x \leq 1.$$

$$a_5 = \frac{\frac{1}{2}(\frac{1}{2} - 1) \ldots (\frac{1}{2} - 4)}{5!} = \frac{7}{256} .$$

Setting $x = \frac{1}{2}$ gives

$$\sqrt{\frac{3}{2}} \approx 1 + \frac{1}{4} - \frac{1}{32} + \frac{1}{128} - \frac{5}{2048} \approx 1.2241$$

with error

$$\frac{7}{256} \cdot \left(\frac{1}{2}\right)^5 = \frac{7}{8192} < 0.001.$$

Thus, correct to two decimal places

$$\sqrt{\frac{3}{2}} \approx 1.22.$$

Exercises 6-7

1. Using (5) show that

$$|\log_e 2 - p_n(1)| \leq 0.005$$

if $n \geq 199$. How large must n be in order that

$$|\log_e 2 - p_n(1)| \leq 5 \times 10^{-10}?$$

2. Estimate $\log_e 1.2$ correct to two decimal places.

3. How large must n be in order to use the Taylor approximation to find $\log_e 0.9$ correct to one decimal place. (Hint: $\log_e 0.9 = -\log_e \frac{10}{9} = -\log_e (1 + \frac{1}{9})$.)

4. (a) Use the Taylor approximation with $n = 5$ to estimate $\log_e 3$.

   (b) What does (5) give as the maximum error in this case?

   (c) Compare your result with the value of $\log_e 3$ in the tables.

   (d) Now use $n = 6, 7, 8, 9$.

   (e) What do you think happens to $\log_e 3 - p_n(2)$ as n becomes large?

5. Find

   (a) $\lim\limits_{x \to 0} \dfrac{\log_e(1 + x)}{x}$

   (b) $\lim\limits_{x \to 0} \dfrac{(\sin x)(\log_e(1 + x))}{(1 - \cos x)}$

6. Find the Taylor approximation of degree 5 to $x \to \sqrt{1 + x}$. Use (8) to estimate R_5 for $0 \leq x \leq 1$.

7. Use the Taylor approximation to $x \to \sqrt{1 + x}$ with $n = 4$ to estimate $\sqrt{2}$. What is the maximum error? Repeat for $n = 5$. (See No. 6.)

8. Use the Taylor approximation to $x \to \sqrt{1 + x}$ with $n = 3$ to estimate $\sqrt{1.1}$. What is the maximum error? Repeat for $n = 4$.

9. Use the Taylor approximation to $x \to \sqrt{1 + x}$ with $n = 4$ to estimate $\sqrt{\frac{1}{2}}$. Compare your result with the estimate

$$\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} = 0.707.$$

10. Find

(a) $\lim_{x \to 0} \frac{\cos x - \sqrt{1 + x}}{\log_e (1 + x)}$

(b) $\lim_{x \to 0} \frac{e^{x^2} - \sqrt{1 + x^2}}{(\sin x)^2}$

11. Find the Taylor approximation of degree three to each of the following:

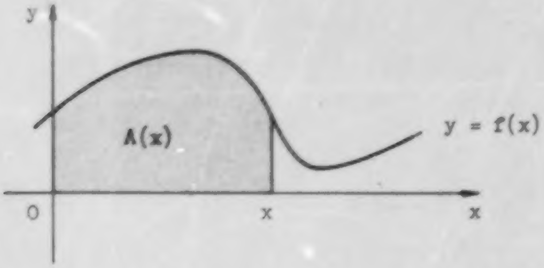
(a) $x \to \sqrt[3]{1 + x}$

(b) $x \to (1 + x)^{5/3}$

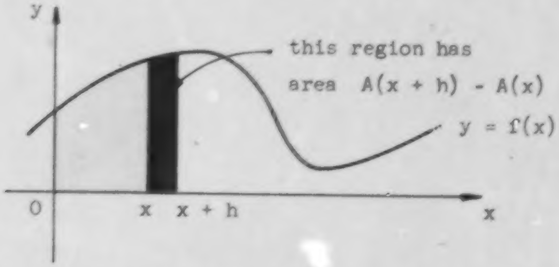
Chapter 7

AREA AND THE INTEGRAL

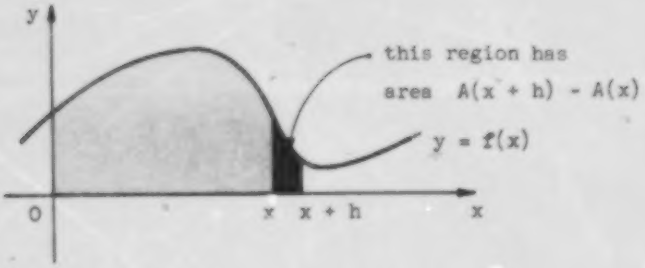
This chapter begins a discussion of the concept of area of a region bounded by the graph of a function. At first glance, the idea of area appears to be entirely unrelated to our discussions of derivatives in Chapters 2, 4, and 6. Upon closer inspection, however, we shall discover that these two ideas must be related. Suppose $A(x)$ represents the area of the shaded region shown in the following figure.



As we move x along the horizontal axis, the area $A(x)$ of the shaded region changes. A measure of the rate of change in $A(x)$ is $A'(x)$, the value of the derivative of the area function at x. The change in area is also related to the height of the graph of f at x, that is, to the value $f(x)$. Consider for example, the case when $f(x)$ is large.



If we move a small amount, say h units, to the right, the area increases fairly quickly, so that the additional area $A(x + h) - A(x)$ is fairly large. If, however, $f(x)$ is close to the x-axis



then the additional area $A(x + h) - A(x)$ will be fairly small.

These considerations lead us to suspect that there must be some relationship between the rate of change of the area function $x \to A(x)$ and the values of f, that is $A'(x)$ must be related to $f(x)$. In this chapter we shall show that for the functions of interest to us in this text, the derivative A' of the area function is f; that is, $A'(x) = f(x)$.

Of course, it is not immediately obvious what the area bounded by a graph should be, particularly if f is not a constant or linear function. Therefore, in the first section, after considering constant and linear cases, we deal with an approximation procedure for obtaining the area of a region bounded by the graph of a nonlinear function (Section 7-1). A proof of the relation $A'(x) = f(x)$ is given in Section 7-2, and extended in Section 7-3 to establish the so-called Fundamental Theorem of Calculus, with the geometric interpretation that the area bounded by the graph of f, the x-axis and vertical lines at a and b is given by the difference $F(b) - F(a)$ where F is any antiderivative of f (that is, $F' = f$). Further notation and properties are introduced in Section 7-4, and the results are extended to signed area in Section 7-5.

The final section discusses the use of antiderivative formulas in calculating areas. Further antidifferentiation methods are discussed in Chapter 9 and Appendix 4.

7-1. Area Under a Graph

We first attack the general problem of finding the area of a region located in the first quadrant, bounded by the graph of a nonnegative function f, the x-axis, the y-axis and a second vertical line, as in Figure 7-1a. We shall not specify the value of the coordinate x at which the second vertical line cuts the x-axis. This will allow us to find general formulas rather than particular numbers. We shall denote the desired area by $A(x)$.

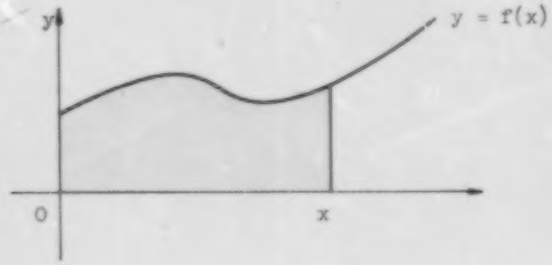


Figure 7-1a

Area under a graph

Frequently the first step a mathematician takes in attacking a new problem is to investigate a few special cases of the problem. He often finds this initial investigation very helpful in setting his mind working towards a general solution. In this spirit we begin with the simplest of polynomial functions and examine the area under the graph of the constant function

$$f : x \to c,$$

where c is a fixed positive number. This case is very easy to handle. In fact, since we know that the area of a rectangle is equal to the product of its base and its height, we see that the desired area is

$$A(x) = cx.$$

(See Figure 7-1b.)

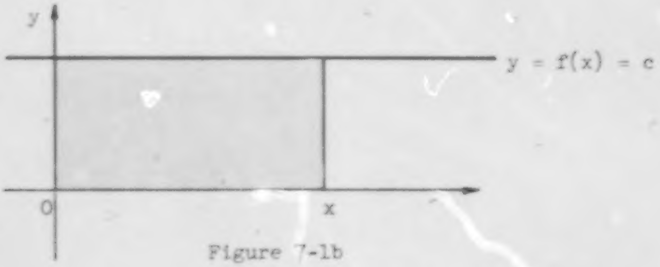


Figure 7-1b

The area of the shaded region is cx.

Note that the "area function"

$$A : x \to cx$$

is a linear function whose derivative A' is

$$f : x \to c.$$

The next case we examine is that of a linear function

$$f : x \to mx + b.$$

The area we wish to find is that of the shaded region in Figure 7-1c.

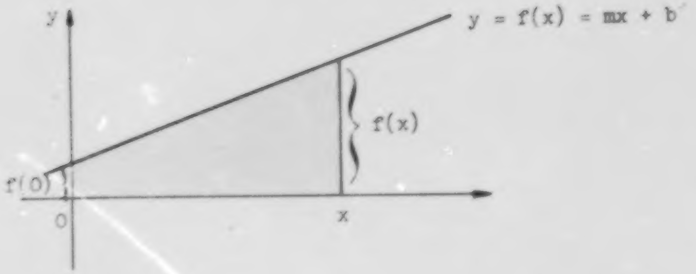


Figure 7-1c

Area under $f : x \to mx + b$

This case is also easy to handle since the shaded region is a trapezoid. We recall that the area of a trapezoid is $\frac{1}{2}$ the sum of the parallel bases times the height. In Figure 7-1c the trapezoid is lying on its side, its "bases" have lengths $f(0)$ and $f(x)$ and its "height" is x. Therefore, the desired area is

$$\begin{aligned} A(x) &= \frac{f(0) + f(x)}{2} \cdot x \\ &= \frac{(m \cdot 0 + b) + (mx + b)}{2} \cdot x \\ &= \frac{mx + 2b}{2} \cdot x \\ &= \frac{mx^2}{2} + bx. \end{aligned}$$

We observe that the derivative A' of the "area function"

$$A : x \to \frac{mx^2}{2} + bx$$

is the linear function

$$f : x \to mx + b.$$

After the constant functions and the linear functions, the next simplest polynomial functions are the quadratic functions. Even though these functions seem to be but a step removed from the linear functions, we shall see that they introduce an entirely new order of complexity. The reason for this is that the graphs of quadratic functions are curves, and we have no formulas for calculating areas of regions bounded by curves (except, of course, when the curves are circles). Hence, it will be wise to move more slowly, and first study a very special case--say the function $f : x \to x^2$. (See Figure 7-1d.)

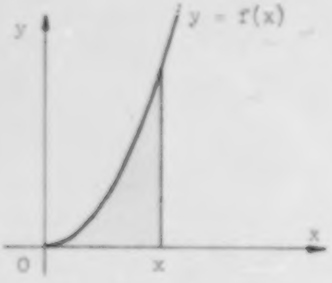


Figure 7-1d

Area under $f : x \to x^2$.

If it were possible to cut the region up into a finite number of rectangular or triangular parts we could add the areas of the parts to obtain the total area. Of course, we cannot do this. The best we can do with such a method is to approximate the area. We can cover the region with rectangles and obtain as the sum of their areas a value that is somewhat larger than the one we seek. On the other hand, we can pack rectangles into the region without overlapping, and obtain in the sum of their areas a value that is somewhat too small. In this way we may at least hope to arrive at an approximate value that we might be able to use in constructing our area function.

Our procedure is to subdivide the line segment from 0 to x into a large number of equal parts, then to use the subintervals as bases of rectangles interior and exterior to the region. To illustrate this procedure we examine a case where the number of subdivisions is small.

Suppose we divide the line segment from 0 to x into 5 equal subintervals. Each of these subintervals will be the base of an interior rectangle, the largest rectangle that can be drawn under the curve with this subinterval as base (Figure 7-1e). Each of these subintervals will also be the base of an exterior rectangle, the smallest rectangle that can be drawn above the curve with this rectangle as base (Figure 7-1f).

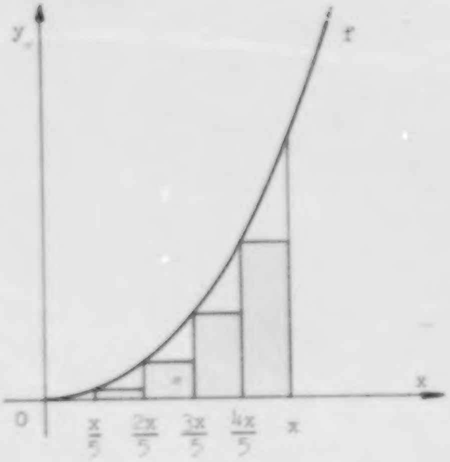


Figure 7-1e

Area approximated by interior rectangles.

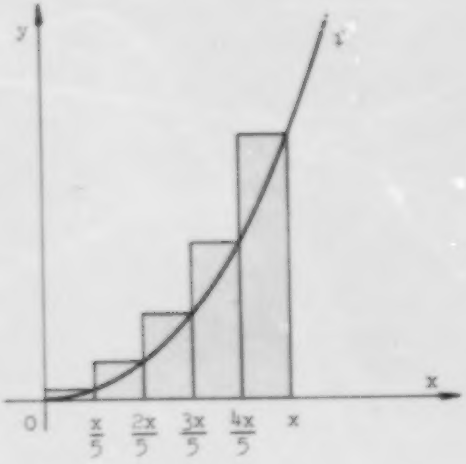


Figure 7-1f

Area approximated by exterior rectangles.

We see from these figures that our desired area $A(x)$ satisfies the two inequalities

(1) $A(x) >$ the sum of the areas of the interior rectangles,

(2) $A(x) <$ the sum of the areas of the exterior rectangles.

Let us calculate the sums of the areas of the interior and exterior rectangles. If we split the segment from 0 to x into 5 equal parts, the length of each part will be $\frac{x}{5}$ and the endpoints of the parts will be

(3) $$0, \frac{x}{5}, \frac{2x}{5}, \frac{3x}{5}, \frac{4x}{5}, \frac{5x}{5}.$$

From Figure 7-1g we see that the height of an interior rectangle is $f(a)$, where a is the left endpoint of its base; the height of an exterior rectangle is $f(b)$, where b is the right endpoint of its base.

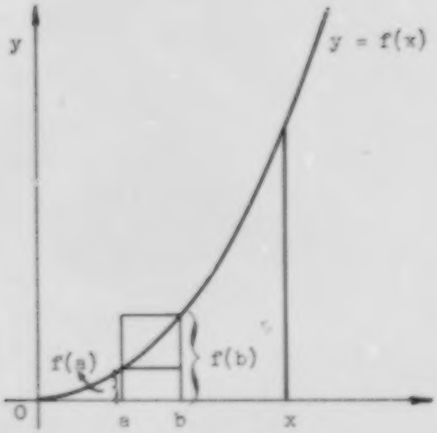


Figure 7-1g

Heights of interior and exterior rectangles.

Using the subdivisions (3) we know that the heights of the (five*) interior rectangles are

$$f(0),\ f(\tfrac{x}{5}),\ f(\tfrac{2x}{5}),\ f(\tfrac{3x}{5}),\ f(\tfrac{4x}{5});$$

the heights of the corresponding exterior rectangles are

$$f(\tfrac{x}{5}),\ f(\tfrac{2x}{5}),\ f(\tfrac{3x}{5}),\ f(\tfrac{4x}{5}),\ f(\tfrac{5x}{5}).$$

Multiplying each of these heights by the common base length $\frac{x}{5}$, we obtain the area of the corresponding rectangles. The sum of the area of the interior rectangles is

$$\tfrac{x}{5}[f(0) + f(\tfrac{x}{5}) + f(\tfrac{2x}{5}) + f(\tfrac{3x}{5}) + f(\tfrac{4x}{5})].$$

The sum of the areas of the exterior rectangles is

$$\tfrac{x}{5}[f(\tfrac{x}{5}) + f(\tfrac{2x}{5}) + f(\tfrac{3x}{5}) + f(\tfrac{4x}{5}) + f(\tfrac{5x}{5})].$$

Since $f : x \to x^2$ we have

---

*The leftmost "rectangular region" has zero area.

$$f(0) = 0,\ f(\tfrac{x}{5}) = \frac{x^2}{25},\ f(\tfrac{2x}{5}) = \frac{4x^2}{25},\ f(\tfrac{3x}{5}) = \frac{9x^2}{25},$$

$$f(\tfrac{4x}{5}) = \frac{16x^2}{25} \text{ and } f(\tfrac{5x}{5}) = \frac{25x^2}{25}.$$

The sum of the areas of the interior rectangles

$$= \frac{x}{5}\left[0 + \frac{x^2}{25} + \frac{4x^2}{25} + \frac{9x^2}{25} + \frac{16x^2}{25}\right]$$

$$= \frac{x^3}{5}\left[\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right]$$

$$= \frac{6x^3}{25}.$$

The sum of the areas of the exterior rectangles

$$= \frac{x^3}{5}\left[\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{25}{25}\right]$$

$$= \frac{11x^3}{25}.$$

Our desired area $A(x)$ lies between these two quantities; that is,

$$\frac{6x^3}{25} < A(x) < \frac{11x^3}{25}.$$

This is certainly not a very accurate estimate of our desired area. If, however, we use a larger number of subdivisions we may hope to improve our estimate.

To obtain a general estimation formula, we let n denote the number of subdivisions of the segment from 0 to x. The length of each part will be $\frac{x}{n}$ and the endpoints will be

$$0,\ \frac{x}{n},\ 2(\tfrac{x}{n}),\ 3(\tfrac{x}{n}),\ \ldots,\ (n-1)(\tfrac{x}{n}),\ n(\tfrac{x}{n}).$$

The heights of the interior rectangles will be

$$f(0),\ f(\tfrac{x}{n}),\ f(\tfrac{2x}{n}),\ \ldots,\ f(\tfrac{(n-1)x}{n}).$$

The heights of the exterior rectangles will be

$$f(\tfrac{x}{n}),\ f(\tfrac{2x}{n}),\ \ldots,\ f(\tfrac{nx}{n}).$$

The sums of the areas of the interior and exterior rectangles will be, respectively

(4) $$\frac{x}{n}\left[f(0) + f(\tfrac{x}{n}) + f(\tfrac{2x}{n}) + \dots + f(\tfrac{(n-1)x}{n})\right]$$

and

(5) $$\frac{x}{n}\left[f(\tfrac{x}{n}) + f(\tfrac{2x}{n}) + \dots + f(\tfrac{nx}{n})\right].$$

Since $f : x \to x^2$, we have

$$f(0) = 0,\ f(\tfrac{x}{n}) = \frac{x^2}{n^2},\ f(\tfrac{2x}{n}) = \frac{4x^2}{n^2}$$

and, in general

$$f(\tfrac{kx}{n}) = \frac{k^2x^2}{n^2};\ k = 0, 1, 2, \dots, n.$$

The interior sum (4) can then be rewritten as

$$\frac{x}{n}\left[0 + \frac{x^2}{n^2} + \frac{4x^2}{n^2} + \dots + \frac{(n-1)^2x^2}{n^2}\right] = \frac{x^3}{n^3}\left[0 + 1 + 4 + \dots + (n-1)^2\right]$$

To simplify this we use the formula* for the first $(n - 1)$ squares

$$1 + 4 + \dots + (n-1)^2 = \tfrac{1}{6}(n-1)(n)(2n-1) = n^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right).$$

We can thus rewrite the interior sum (4) as

$$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

A similar process applied to the exterior sum (5) gives the sum of the areas of the exterior rectangles

$$\frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

Our desired area $A(x)$ lies between these two quantities; that is,

(6) $$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2} < A(x) < \frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}.$$

---

*See Appendix 3.

This must be true for each positive integer n. If x is fixed and n is very large compared to x each of the terms

$$-\frac{x^3}{2n}, \frac{x^3}{2n} \text{ and } \frac{x^3}{6n^2}$$

must be very close to zero. This process suggests that the only value that the area $A(x)$ can have is $\frac{x^3}{3}$.

We summarize: if $f : x \to x^2$ and $A(x)$ is the area of the region bounded by the x-axis, the y-axis, the graph of f and the vertical line x units to the right of the origin, then

$$A : x \to \frac{x^3}{3}.$$

Note that the derivative of the area function is

$$f : x \to \frac{3x^2}{3} = x^2;$$

that is, $A' = f$.

This same relationship $A' = f$ was true in the case of constant and linear functions. We might conjecture that it is always true. In Section 7-2 we shall show that it is indeed true for a wide class of functions f, a class which includes most of the functions of interest to us in this book.

Exercises 7-1

1. We showed in this section that the region bounded by the coordinate axes, $y = x^2$, and a vertical line at $\underline{x}$, has an area which is between the sum of the interior and the exterior rectangles. This inequality (6) was

$$x^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(x) < x^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

(a) It follows that

$$1^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(1) < 1^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

Express this relationship when

(i) $n = 5$
(ii) $n = 100$

(b) From (6) we know that

$$2^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(2) < 2^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).$$

Using directly the results of part (a), i.e., with minimum computation, express this relationship when

(i) $n = 5$
(ii) $n = 100$

(c) Using $A : x \rightarrow \frac{1}{3}x^3$ for the area function associated with the function, $f : x \rightarrow x^2$, find the area in the first quadrant of the region bounded by the coordinate axes, $y = x^2$, and the vertical line at

(i) $x = \frac{1}{2}$

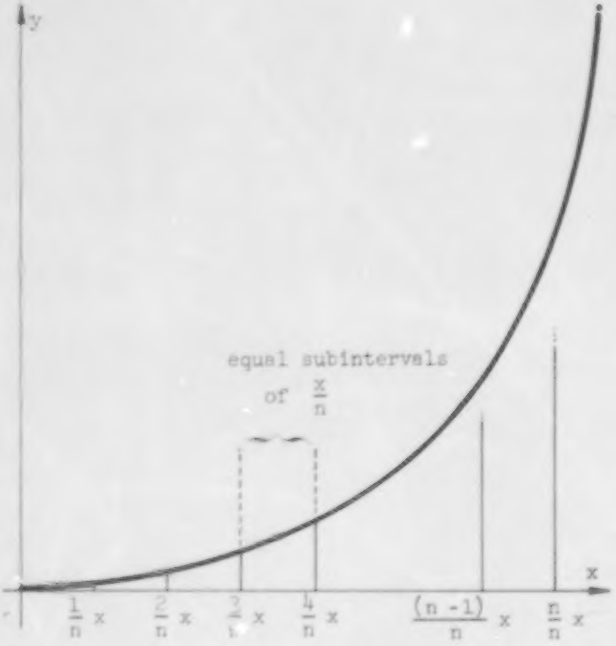
(ii) $x = \sqrt[3]{3}$

2. If $f : x \to x^3$, and $A(x)$ is the area of the region depicted in the sketch to the right, show that the area function is

$$A : x \to \frac{x^4}{4},$$

using the method of this section for finding the area function of $x \to x^2$.

[Hint: The sum of the cubes of the first $n - 1$ integers is $\left(\frac{(n-1)n}{2}\right)^2$.]



Equal sub-intervals of $\frac{x}{n}$.

(a) First, show that the sum of the areas of the interior rectangles is

$$\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right).$$

(b) Second, find the sum of the areas of the exterior rectangles, showing that

$$\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(x) < \frac{x^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right),$$

and as $n \to \infty$, $A : x \to \frac{1}{4}x^4$.

(c) Next, using the inequality of part (b) above, and letting $x = 1$, find an expression for $A(1)$, when

(i) $n = 5$
(ii) $n = 100$

(d) From the expressions found for $A(1)$ in part (c) above, find, with minimum computation, an expression for $A(2)$, when $n = 100$.

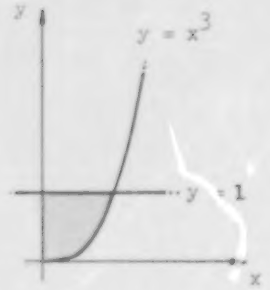
(e) Using $A : x \to \frac{1}{4} x^4$ for the area function associated with the function, $f : x \to x^3$, find the area in the first quadrant of the region bounded by the coordinate axes, $y = x^3$, and the vertical line at

(i) $x = 0.4$

(ii) $5\sqrt{2}$

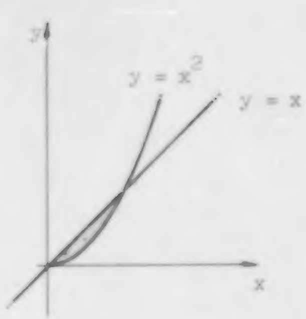
3. Find the area of the region in the first quadrant bounded by $x = 0$, $y = 1$, and $y = x^3$.

[Hint: $y = 1$ and $y = x^3$ intersect at $(1,1)$. The shaded area equals the area under $y = 1$ minus the area under $y = x^3$ (between $x = 0$ and $x = 1$).]



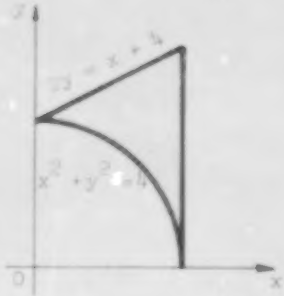
4. Find the area of the region in the first quadrant bounded by $y = x$ and $y = x^2$.

[Hint: Find the intersection points; find the area under each curve between intersection points; find the difference between these areas.]



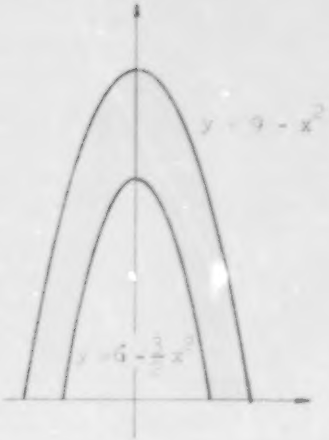
5. Sketch $y = x^3$ and $y = x^2$. $[-\frac{1}{2} < x < \frac{3}{2}]$
In a similar manner to that of Number 3 and Number 4, find the area between the two curves.

6. Find the area in quadrant one bounded by the quarter circle (with center at origin and radius 2), the line $x - 2y + 4 = 0$, and the vertical line tangent to the circle.
[Hint: Find intersection points; find area of quarter circle by geometry; subtract areas.]



7. Find area of region bounded by $y = 0$, $y = 9 - x^2$, and $y = 6 - \frac{3}{2}x^2$.

[Hint: Use symmetry.]



8. (a) For the function $f : x \to x^2$, we developed in this section an inequality for the area function:

$$\frac{x^3}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right) < A(x) < \frac{x^3}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right)$$

Show that if we average these sums of areas of interior and exterior for $n = 5$, we have $A(x) \approx \frac{17}{50}x^3$.

(b) Now estimate $A(x)$ for the same function by connecting $(0, f(0))$ to $\left(\frac{x}{5}, f(\frac{x}{5})\right)$, $\left(\frac{x}{5}, f(\frac{x}{5})\right)$ to $\left(\frac{2x}{5}, f(\frac{2x}{5})\right)$, ..., and summing the resulting trapezoids.

(c) As a third estimate, sum 5 rectangles with equal widths along the the x-axis, and heights erected at the midpoint of each interval; i.e., the width of each rectangle would be $\frac{x}{5}$, and the heights would be $\frac{x}{10}$, $\frac{3x}{10}$, ...

(d) Which of these three estimates above is the closest to the exact area of $\frac{1}{3}x^3$.

7-2. The Area Theorem

In Section 7-1 we found some formulas for the area of the region in the first quadrant bounded by the graph of a function f, the x-axis, the y-axis and a second vertical line, x units to the right of the origin, such as that shown in Figure 7-2a.

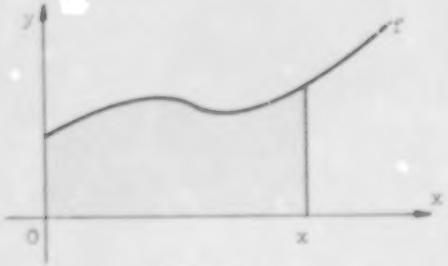


Figure 7-2a

Area Under a Graph

Calling the indicated area $A(x)$ we obtained a function $x \to A(x)$, which we called the "area function." The results obtained in Section 7-1 can be tabulated as follows:

Function f	Area function A	Derivative of area function A'
$x \to c$	$x \to cx$	$x \to c$
$x \to mx + b$	$x \to \frac{mx^2}{2} + bx$	$x \to mx + b$
$x \to x$	$x \to \frac{x^3}{3}$	$x \to x^2$

It is impossible to miss the similarity between the first and third columns of this table. Since these two columns are identical except for heading we are practically compelled to suspect that there must be some relationship between f and the derivative A' of its area function A. We conjecture:

(1) If A is the area function associated with a function f, then $A' = f$.

We shall prove this result with the following assumptions on f

(a) f is an increasing function; that is,

$$f(c) < f(d) \quad \text{if} \quad 0 \leq c < d. \tag{2}$$

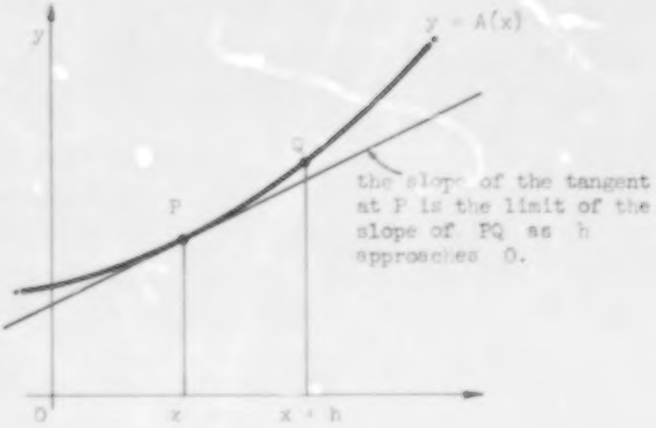
(b) The graph of f has no "gaps" for $x \geq 0$.

Condition (b) means that if $x \geq 0$, $\lim_{h \to 0} f(x + h) = f(x)$. When condition (b) is satisfied we say that f is continuous for $x \geq 0$.

To prove (1) we must show that

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x);$$

that is, that the slope of the line through $P(x, A(x))$ and $Q(x + h, A(x + h))$ approaches $f(x)$ as h approaches 0. Since the indicated limit is just $A'(x)$, which is the slope of the tangent line at $P(x, A(x))$, we shall then know that $A'(x) = f(x)$. (See Figure 7-2b.)



the slope of the tangent at P is the limit of the slope of PQ as h approaches 0.

Figure 7-2b

Graph of the Area Function

Let us first suppose that $h > 0$, so that the graph of f is something like that shown in Figure 7-2c. The two quantities $A(x)$ and $A(x + h)$ are the areas of the regions bounded by the y-axis, the x-axis, the graph of f and the vertical lines which are respectively x and $x + h$ units to the right of the origin. Hence, the difference

$$A(x + h) - A(x)$$

represents the area of the shaded region shown in Figure 7-2c.

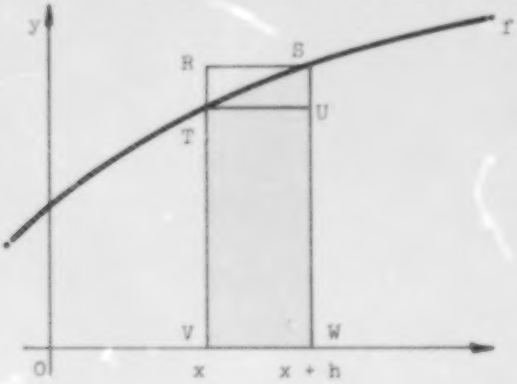


Figure 7-2c

$A(x + h) - A(x) =$ Area of the shaded region

Since we have assumed that f is increasing, the shaded region of Figure 7-2c includes the smaller rectangle TUWV and is included in the larger rectangle RSWV. These rectangles have base length h and the respective heights $f(x)$ and $f(x + h)$. Thus

$$hf(x) < \text{area of shaded region} < hf(x + h);$$

that is,

$$hf(x) < A(x + h) - A(x) < hf(x + h).$$

This inequality used the assumption that $h > 0$. If we divide by h we obtain

$$(4) \qquad f(x) < \frac{A(x + h) - A(x)}{h} < f(x + h).$$

From (3) if h approaches 0 then $f(x + h)$ approaches $f(x)$. Hence, if h is positive and h approaches 0

$$\frac{A(x + h) - A(x)}{h} \text{ approaches } f(x).$$

Comparable arguments will give the same result if $h < 0$, so that, indeed $A'(x) = \lim\limits_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x)$, that is, $A' = f$, if the assumptions (2) hold. We can, of course, replace the assumption that f is increasing by the assumption that f is decreasing. This will invert the inequality signs in (4) but will not change the conclusion.

In the above proof we used the fact that

$$A(x + h) - A(x)$$

is the area of the shaded region shown in Figure 7-2c. This will also be true if the lower limit is taken to be any number $a \leq x$. In other words we can let $A(x)$ represent the area of the shaded region shown in Figure 7-2d. The difference

$$A(x + h) - A(x)$$

will be the area of the darkly shaded region shown in Figure 7-2e.

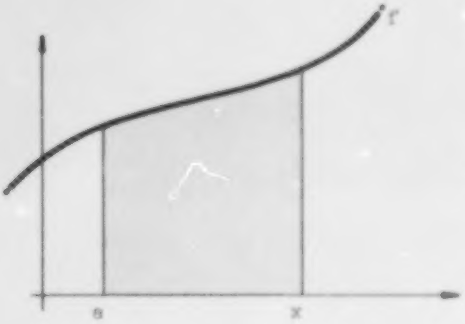


Figure 7-2d

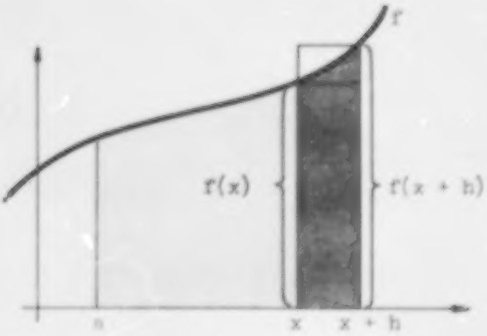


Figure 7-2e

Assuming that f is increasing for $x > a$ we could repeat the foregoing arguments to conclude that

$$f(x) < \frac{A(x + h) - A(x)}{h} < f(x + h), \quad \text{if } h > 0$$

and

$$f(x) > \frac{A(x + h) - A(x)}{h} > f(x + h), \quad \text{if } h < 0.$$

If we assume that the graph of f is continuous, then $f(x + h)$ approaches $f(x)$ and

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x)$$

as h approaches 0. Hence, $A' = f$.

This fact that the derivative of the area function is f will be referred to as the Area Theorem.[*]

AREA THEOREM. Suppose f is nonnegative and increasing on the interval $a \leq x \leq b$ and that the graph of f has no "gaps." For each x in this interval, if

$$A(x)$$

is the area bounded by the y-axis, the graph of f and ordinates at a and x $(a < x \leq b)$ then

$$A'(x) = f(x).$$

The same result will hold if f is assumed to be decreasing on the interval. In the appendices it will be shown that the theorem remains true under more general conditions.

The Area Theorem doesn't yet tell us how to find the area function $x \to A(x)$. It only tells us that the derivative A' must be f. Consider, for example, the problem of finding the area function A if $f : x \to x^3$.

We know that the derivative of

$$x \to x^4$$

is the function $x \to 4x^3$, so if we divide by 4 then the derivative of

$$x \to \frac{1}{4} x^4 \quad \text{is} \quad x \to x^3.$$

We call $x \to \frac{1}{4} x^4$ an antiderivative of $x \to x^3$. Thus a good candidate for A is

$$A : x \to \frac{1}{4} x^4.$$

Note, however, that the derivative of

$$x \to \frac{1}{4} x^4 + 10$$

is also $x \to x^3$. In fact, if C is any constant then the derivative of

$$x \to \frac{1}{4} x^4 + C \quad \text{is} \quad x \to x^3,$$

---

[*] This is also sometimes known as the Fundamental Theorem of Calculus, a subsequent theorem which can be established analytically without area arguments.

so that any function of the type $x \to \frac{1}{4}x^4 + C$ is a candidate for A. Fortunately, there are no other possibilities for A. This is a consequence of the **following** theorem.

> THE CONSTANT DIFFERENCE THEOREM. If $G'(x) = F'(x)$, $a \leq x \leq b$, then there is a constant C such that
>
> $$G(x) = F(x) + C, \quad a \leq x \leq b.$$

We shall give an intuitive argument. A more complete proof will be found in Chapter 8.

Proof: If $G'(x) = F'(x)$, the graph of F and the graph of G have the same slope at each x on the interval $[a,b]$. This can happen only if either the graphs are the same $(G(x) = F(x))$ or if one graph can be obtained by raising or lowering the other a certain amount $(G(x) = F(x) + C$ for some constant $C)$. (See Figure 7-2f.)

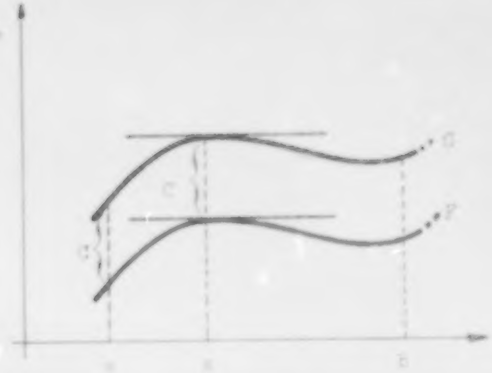


Figure 7-2f

When $x = a$,

$$G(a) = F(a) + C.$$

Therefore, the constant is

$$C = G(a) - F(a).$$

Example 7-2a. Find $A(x)$ if $f : x \rightarrow x^3$.

We know that if $F : x \rightarrow \frac{1}{4} x^4$, then $F' = f$. The Area Theorem tells us that $A' = f$. From the Constant Difference Theorem, since $A' = F'$, there must be a constant C such that

$$A(x) = F(x) + C.$$

To determine C, we need to know $A(x)$ and $F(x)$ for one value of x, say $x = 0$. Since $F(0) = A(0) = 0$

$$0 = 0 + C$$

and $C = 0$. Therefore,

$$A(x) = \frac{1}{4} x^4.$$

Example 7-2b. Find the area between the graph of $f : x \rightarrow x^2 + 2x$, the x-axis and the lines $x = 1$ and $x = 2$.

$$A'(x) = x^2 + 2x.$$

If $F(x) = \frac{x^3}{3} + x^2$, $F'(x) = x^2 + 2x$. By the Constant Difference Theorem

$$A(x) = x^2 + 2x + C$$

for some constant C. Since $A(1) = 0$

$$0 = 1 + 2 + C$$

and $C = -3$ and

$$A(x) = x^2 + 2x - 3.$$

Then $A = A(2) = 4 + 4 - 3 = 5$ is the required area.

We need a notation for the area A of the region bounded by the x-axis, the graph of f and the two vertical lines given by $x = a$ and $x = b$. (See Figure 7-2g.)

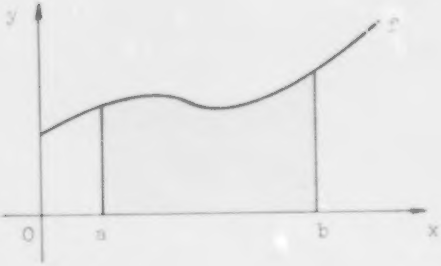


Figure 7-2g

Area under a graph.

The usual symbol for this is

$$\int_a^b f(x)dx$$

suggested by the procedure (described in the previous section) of approximating sums for finding areas. The symbol "$\int$" (a modified letter S) indicates summation. The $f(x)$ is meant to suggest the ordinate of an outer or inner rectangle and the "dx" (an indivisible symbol) the difference in the x's at the ends of the base of a rectangle.

The symbol $\int_a^b f(x)dx$ may be read "The integral of f from a to b." We shall sometimes write this integral more briefly as $\int_a^b f$.

Exercises 7-2

1. In Section 7-1 we obtained the estimates

$$\frac{x^3}{3} - \frac{x^3}{2n} + \frac{x^3}{6n^2} < A(x) < \frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}$$

for each positive integer n, where

$$A(x) = \int_0^x f; \quad f : x \to x^2.$$

Average these to obtain the general estimate

$$A(x) \approx \frac{x^3}{3} + \frac{x^3}{6n^2} .$$

Use this estimate for $A(x)$ in order to calculate approximations of the following quantities when $n = 10$.

(a) $A(2)$

(b) $A(2.1)$

(c) $\dfrac{A(2.1) - A(2)}{0.1}$

(d) $\dfrac{A(x + h) - A(x)}{h}$ for general positive x, h.

(e) Let h approach 0 in (d) and use this to estimate $A'(x)$.

2. Suppose $f : x \to x^2 + 1$. Find

(a) $\displaystyle\lim_{h \to 0} \int_1^{1+h} f(x)dx$

(b) $\displaystyle\lim_{h \to 0} \frac{1}{h} \int_1^{1+h} f(x)dx$

(c) Did you need to calculate $\displaystyle\int_1^{1+h} f(x)dx$ in order to answer (a) and (b)? Explain.

3. Suppose $A(x) = \int_2^x f$, where $f : x \to x^3$

   (a) What is $A(2)$?

   (b) What is $A'(3)$?

   (c) Did you need to find an antiderivative for f in order to answer (a) or (b)?

4. Find two distinct functions g such that g' is the function $x \to 3x^2$. How are your functions related to each other?

5. Find the area bounded by the coordinate axes, the line $x = 2$, and the graph of the function f, where

   (a) $f : x \to x^2$

   (b) $f : x \to 2x + 1$

   (c) $f : x \to 4x^3 + x$

6. (a) Sketch the graph of $f : x \to x^2 + 1$.

   (b) Mark the region bounded by this graph, the coordinate axes, and the line $x = 1$. Find the area of this region.

   (c) Mark the region bounded by your graph, the coordinate axes, and the line $x = 2$. Find the area of this region.

   (d) Mark the region bounded by your graph, the x-axis, and the lines $x = 1$ and $x = 2$. How is this region related to the regions you marked in (b) and (c)? Find its area.

7. (a) Sketch the graph and find the area bounded by the graph of $f : x \to 16 - x^2$, the x-axis, and lines $x = 2$ and $x = 3$.

   (b) Sketch the graph and find the area bounded by the graph of $f : x \to 4x^3 - x$, the x-axis, and the lines $x = 1$ and $x = 2$.

8. For $f : x \to (x - 1)^2$ show how the interval $0 \leq x \leq 3$ can be subdivided so that on each subinterval f is always increasing or always decreasing. Make a sketch.

7-3. The Fundamental Theorem of Calculus

The following theorem summarizes the method for finding area functions explained in the previous section. This theorem is generally referred to as the Fundamental Theorem of Calculus[*], and provides a basic technique for calculating areas by using antiderivatives.

THE FUNDAMENTAL THEOREM OF CALCULUS. If f is nonnegative, increasing and its graph has no gaps on the interval $a \leq x \leq b$, and if F is any function whose derivative is f on this interval, then

$$\int_a^x f = F(x) - F(a), \quad a \leq x \leq b.$$

Proof. The area function

$$A(x) = \int_a^x f$$

is a function whose derivative is f (from the Area Theorem). Furthermore, $A(a) = 0$. Since the functions F and A have the same derivative, f, the Constant Difference Theorem implies that there is a constant C such that

$$A(x) = F(x) + C, \quad a \leq x \leq b.$$

Then

$$C = A(a) - F(a).$$

Since

$$A(a) = 0$$

$$C = -F(a)$$

and

$$A(x) = F(x) - F(a).$$

That is

$$\int_a^x f = F(x) - F(a).$$

Remark. This theorem will still be true if f is assumed to be decreasing on the interval, for the Area Theorem will remain true and the above proof can be repeated verbatim. The theorem is easily extended to the case when the interval can be subdivided into smaller intervals, on each of which f increases or decreases. For example, suppose that $F' = f$ and that f increases for $a \leq x \leq c$ and decreases for $c \leq x \leq b$. (See Figure 7-3a.) Now

---

[*] It relates differentiation and integration.

$$\int_a^b f = A + B = \int_a^c f + \int_c^b f.$$

We apply the Fundamental Theorem to each term and obtain

$$\int_a^c f = F(c) - F(a), \quad \int_c^b f = F(b) - F(c).$$

When we add the two integrals the term $F(c)$ drops out.

We have

$$\int_a^b f = F(c) - F(a) + F(b) - F(c)$$
$$= F(b) - F(a).$$

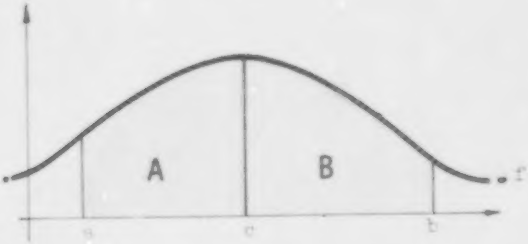


Figure 7-3a

Area of Shaded Region = Area of A + Area of B

When we deal with specific functions we shall use the longer notation for integrals. Thus, instead of writing

$$A = \int_0^1 f \quad \text{where} \quad f : x \to e^x,$$

we shall write simply

$$(1) \qquad A = \int_0^1 e^x \, dx.$$

In this example, since $De^x = e^x$, $F(x) = e^x$ and the Fundamental Theorem tells us that

$$A = e^1 - e^0 = e - 1.$$

Since we can describe f equally well by

$$f : t \rightarrow e^{t}$$

we could replace (1) by

$$(2) \qquad A = \int_{0}^{1} e^{t}\, dt.$$

Because $De^{t} = e^{t}$, $F(t) = e^{t}$ and by the Fundamental Theorem

$$A = e^{1} - e^{0} = e - 1$$

exactly as before. Because the result does not depend on the letter used, the letter x in (1) is called a dummy variable.

Example 7-3a. Find $A(x) = \int_{2}^{x} t^{4}\, dt$. The derivative of $t \rightarrow t^{5}$ is $t \rightarrow 5t^{4}$. Hence the derivative of $F : t \rightarrow \frac{1}{5} t^{5}$ is $f : t \rightarrow t^{4}$. By the Fundamental Theorem

$$A(x) = F(x) - F(2) = \frac{1}{5} x^{5} - \frac{32}{5}.$$

Example 7-3b. $\int_{-\pi/2}^{\pi/2} \cos x\, dx.$

The sine function $F : x \rightarrow \sin x$ is a function whose derivative is f. The interval can be subdivided into two subintervals (namely $-\frac{\pi}{2} \leq x \leq 0$ and $0 \leq x \leq \frac{\pi}{2}$) so that f increases on the first subinterval and decreases on the second interval (see Figure 7-3b). We can, therefore, apply the remark following the Fundamental Theorem to conclude that

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} f &= F(\tfrac{\pi}{2}) - F(-\tfrac{\pi}{2}) \\ &= \sin \tfrac{\pi}{2} - \sin(-\tfrac{\pi}{2}) \\ &= 1 - (-1) \\ &= 2. \end{aligned}$$

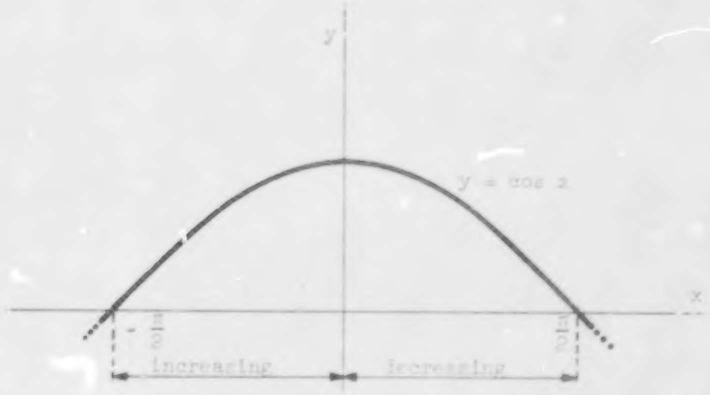


Figure 7-3b

A function F whose derivative is f is called an antiderivative (or indefinite integral) of f. It is also common to use the notation

$$F(x)\Big|_a^b \quad \text{for} \quad F(b) - F(a).$$

The Fundamental Theorem of Calculus may be stated in the form:

$$\boxed{\begin{array}{l}\displaystyle\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a), \\ \text{where } F \text{ is an antiderivative}^{*} \text{ of } f.\end{array}} \tag{3}$$

For example, since the derivative of

$$x \rightarrow \frac{1}{3}x^3 \quad \text{is} \quad x \rightarrow x^2$$

we say that $x \rightarrow \frac{1}{3}x^3$ is an antiderivative of $x \rightarrow x^2$ and write

$$\int_a^b x^2\,dx = \frac{1}{3}x^3\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

---

[*] $F' = f$

Example 7-3c. Find $\int_1^4 t^5\,dt$

First we find an antiderivative of $t \to t^5$. Differentiation of polynomials reduces the degree by one, so antidifferentiation should raise the degree by one. If we recall that the function

$$t \to t^6$$

has the derivative $t \to 6t^5$, we can see that

$$t \to \frac{1}{6} t^6$$

is an antiderivative of $t \to t^5$. Therefore, we have

$$\int_1^4 t^5\,dt = \frac{1}{6} t^6 \Big|_1^4 = \frac{4^6}{6} - \frac{1^6}{6} = \frac{4095}{6}.$$

Example 7-3d. Find the area of the region between the x-axis and one arch of the sine curve given by $y = \sin x$. We want to find (Figure 7-4c).

$$\int_0^\pi \sin x\,dx.$$

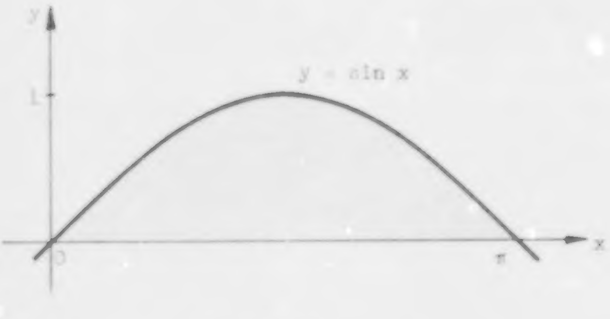


Figure 7-3c

$$\int_0^\pi \sin x\,dx = \text{area of shaded region.}$$

The derivative of the cosine function is the negative of the sine function so that

$$\to -\cos x$$

is an antiderivative of $x \rightarrow \sin x$. We have

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -\cos \pi + \cos 0$$

$$= -(-1) + 1 = 2.$$

Example 7-3e. Find $\int_0^3 (x^2 + 2x + 4)dx$.

We could find an antiderivative of

$$x \rightarrow x^2 + 2x + 4$$

directly and use (3). An alternative approach (which amounts to the same thing) is to remember that the integral of a sum is the sum of the integrals, so that we can write

$$\int_0^3 (x^2 + 2x + 4)dx = \int_0^3 x^2 dx + \int_0^3 2x \, dx + \int_0^3 4dx.$$

The functions

$$x \rightarrow x^2, \; x \rightarrow 2x \quad \text{and} \quad x \rightarrow 4$$

have the respective antiderivatives

$$x \rightarrow \frac{1}{3} x^3, \; x \rightarrow x^2 \quad \text{and} \quad x \rightarrow 4x;$$

so we have

$$\int_0^3 (x^2 + 2x + 4)dx = \frac{1}{3} x^3 \Big|_0^3 + x^2 \Big|_0^3 + 4x \Big|_0^3.$$

$$= \frac{1}{3}(3^3 - 0^3) + (3^2 - 0^2) + (4 \cdot 3 - 4 \cdot 0)$$

$$= 30.$$

Example 7-3f. Describe the area of the region between the graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ as the difference of two integrals and evaluate.

The area of region A in Figure 7-3d is

$$\int_0^1 \sqrt[3]{x}\, dx - \int_0^1 \sqrt{x}\, dx.$$

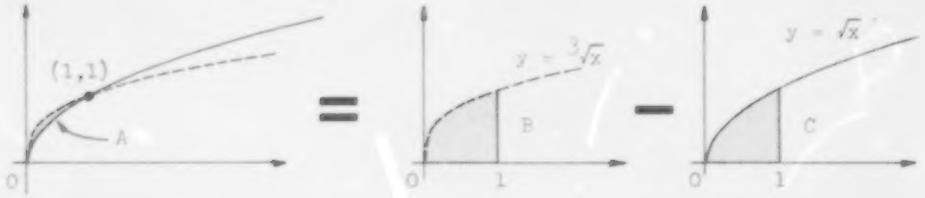


Figure 7-3d

Area of A = Area of B - Area of C

To find antiderivatives of $x \to \sqrt[3]{x}$ and $x \to \sqrt{x}$, we first write $\sqrt[3]{x} = x^{1/3}$ and $\sqrt{x} = x^{1/2}$ and then recall the power formula

$$Dx^a = ax^{a-1}.$$

Here differentiation amounts to multiplying by the exponent and reducing the exponent by 1. Then antidifferentiation amounts to raising the exponent by 1 and dividing by the new exponent. Thus, we have

$$x \to \frac{3}{4} x^{4/3} \quad \text{and} \quad x \to \frac{2}{3} x^{3/2}$$

as respective antiderivatives of $x \to \sqrt[3]{x}$ and $x \to \sqrt{x}$. Therefore, our desired area is

$$\begin{aligned}\int_0^1 \sqrt[3]{x}\, dx - \int_0^1 \sqrt{x}\, dx &= \frac{3}{4} x^{4/3} \Big|_0^1 - \frac{2}{3} x^{3/2} \Big|_0^1 \\ &= \frac{3}{4} - \frac{2}{3} \\ &= \frac{1}{12}.\end{aligned}$$

Example 7-3g. Evaluate $\int_{-\pi/2}^{\pi} |\sin x|dx.$

In Figure 7-3e we indicate (by shading) the region whose area is the integral we wish to evaluate.

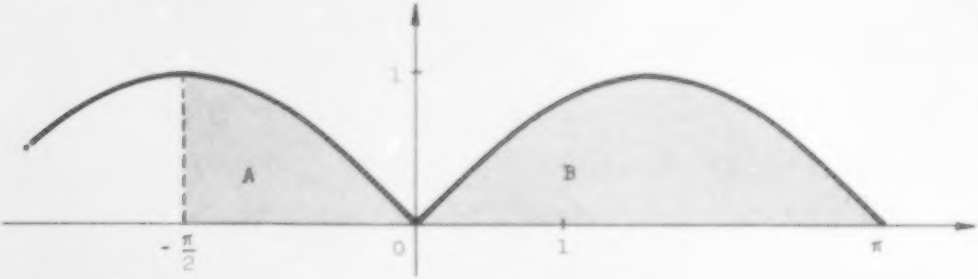


Figure 7-3e

We know that the area of region B is 2 (from Example 7-3d) and we should suspect that the total area of regions A and B is 3. We can confirm this suspicion and gain additional experience using antiderivatives. By definition of absolute value we have

$$x \to |\sin x| = \left\{\begin{array}{ll} \sin x, & \text{for } \sin x \geq 0 \\ -\sin x, & \text{for } \sin x < 0 \end{array}\right\}.$$

We express our integral as the sum of two integrals:

$$(4) \qquad \int_{-\pi/2}^{\pi} |\sin x|dx = \int_{-\pi/2}^{0} |\sin x|dx + \int_{0}^{\pi} |\sin x|dx$$

$$= \int_{-\pi/2}^{0} (-\sin x)dx + \int_{0}^{\pi} \sin x \, dx.$$

Antiderivatives of

$$x \to -\sin x \quad \text{and} \quad x \to \sin x$$

are, respectively,

$$x \to \cos x \quad \text{and} \quad x \to -\cos x.$$

Therefore, we have

$$\int_{-\pi/2}^{\pi} |\sin x|dx = \cos x \Big|_{-\pi/2}^{0} + (-\cos x) \Big|_{0}^{\pi}$$

$$= \cos 0 - \cos(-\tfrac{\pi}{2}) + (-\cos \pi) - (-\cos 0)$$

$$= 1 - 0 + (-(-1)) - (-1)$$

$$= 3.$$

Example 7-3h. Evaluate $\int_0^2 f(x)dx$ if

$$f(x) = \begin{cases} \sqrt{3x}, & \text{for } 0 \le x \le 1 \\ (2x - 1)^2, & \text{for } 1 < x \le 2 \end{cases}.$$

The area of the shaded region in Figure 7-3f is given by the integral we wish to evaluate. Note the break in the graph of f at $x = 1$. In order to be able to apply the Fundamental Theorem of Calculus, we first break our interval into subintervals over which the graph of f has no gaps:

$$\int_0^2 f(x)dx = \int_0^1 \sqrt{3x}\, dx + \int_1^2 (2x - 1)^2 dx.$$

Antiderivatives for $x \to \sqrt{3x}$ and $x \to (2x - 1)^2$ are respectively

$$x \to \frac{2\sqrt{3}}{3} x^{3/2} \quad \text{and} \quad x \to (\tfrac{1}{2})(\tfrac{1}{3})(2x - 1)^3.$$

(Check by differentiation and see Exercises 7-3, No. 5). We therefore have

$$\int_0^2 f(x)dx = \frac{2\sqrt{3}}{3} x^{3/2} \Big|_0^1 + \frac{(2x - 1)^3}{6} \Big|_1^2$$

$$= \frac{2\sqrt{3}}{3}(1^{3/2} - 0^{3/2}) + \frac{1}{6}(3^3 - 1^3)$$

$$= \frac{2\sqrt{3} + 13}{3}.$$

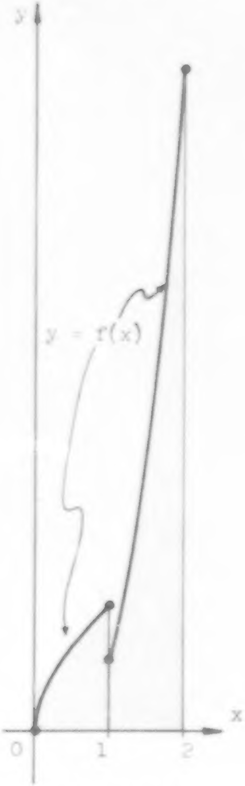


Figure 7-3f

Exercises 7-3

1. Find each of the following integrals.

   (a) $\int_{1}^{2} (x^2 + x + 3)dx$

   (b) $\int_{-2}^{0} (x^2 + x + 3)dx$

   (c) $\int_{-2}^{2} (x^2 + x + 3)dx$

   (d) $\int_{0}^{\pi/3} \cos x \, dx$

   (e) $\int_{0}^{2} \sqrt{x^3} \, dx$

   (f) $\int_{1/16}^{1} (\sqrt{x} + \sqrt[4]{x})dx$

   (g) $\int_{1/2}^{1} \frac{1}{3x^2} \, dx$

   (h) $\int_{-2}^{-1} (5x^{-6} + x^2)dx$

   (i) $\int_{1}^{2} \frac{1}{x} \, dx$

   (j) $\int_{1}^{\pi} x^n \, dx$

   (k) $\int_{-1}^{2} e^x \, dx$

   (ℓ) $\int_{-1}^{2} (e^x + 1)dx$

   (m) $\int_{-1}^{2} (e^x + x)dx$

   (n) $\int_{1}^{2} (5x^4 + 3x^2 + 1)dx$

   (o) $\int_{\pi/6}^{\pi/3} (\sin x + \cos x)dx$

   (p) $\int_{0}^{4\pi/3} (e^x + \sin x)dx$

   (q) $\int_{3}^{3} (x^2 + 2x + 5)dx$

   (r) $\int_{10}^{10} \tan x \, dx$

2. Sketch the regions bounded by the x-axis, the curve $y = f(x)$ and the vertical lines $x = a$ and $x = b$. Then find the areas

   (a) $f : x \to x^3 + 2x + 1, \quad a = 1, \quad b = 3$

   (b) $f : x \to e^x, \quad a = -1, \quad b = 1$

   (c) $f : x \to e^x + x^2, \quad a = -1, \quad b = 1$

   (d) $f : x \to \sin x + \cos x, \quad a = 0, \quad b = \frac{\pi}{2}$

   (e) $f : x \to 2x^4 + \cos x, \quad a = -\frac{\pi}{2}, \quad b = \frac{\pi}{4}$

   (f) $f : x \to x^{-10}, \quad a = -1, \quad b = -\frac{1}{2}$

   (g) $f : x \to \sqrt[3]{x^2}, \quad a = -1, \quad b = 1$

3. Sketch the region bounded by the x-axis, $y = f(x)$ and the given vertical lines; then find its area.

   (a) $f : x \to |x|$; vertical lines $x = -2$, $x = 4$
   (Check your result by elementary geometry.)

   (b) $f : x \to |4x^3|$; vertical lines at $x = -1$, $x = 3$

   (c) $f : x \to |\cos x|$; vertical lines at $x = -\frac{\pi}{3}$, $x = \frac{4\pi}{3}$

   (d) $f : x \to |\frac{1}{2} - \sin x|$; vertical lines at $x = -\pi$, $x = 2\pi$

   (e) $f : x \to |1 - \sqrt{x}|$; vertical lines at $x = 0$, $x = 4$

4. (a) Evaluate $(x^2 + 3\sqrt{x}) \Big|_1^4$ and $(x^2 + 3\sqrt{x} + 50) \Big|_1^4$

   (b) Suppose $F(x) = G(x) + \log_e 2$ where $F(0) = 1$, $F(1) = -1$.
   Find $G(x) \Big|_0^1$.

   (c) What is $F(x) \Big|_a^b - G(x) \Big|_a^b$ if $F' = G'$?

5. (a) Find an antiderivative for each of the following functions.

   (i) $f : x \to (x - 1)^3$

   (ii) $F : x \to x^3 - 3x^2 + 3x - 1$

   (iii) $g : x \to 8x^3 - 12x^2 + 6x - 1$

   (iv) $G : x \to (2x - 1)^3$
   [Hint: Put G in the form $a(x - b)^n$.]

   (b) Compare the functions F with f and G with g. Compare the antiderivatives.

6. Find an antiderivative for each of the following functions

$$f : x \to 3(x + 1)^2$$
$$g : x \to (3x + 2)^5.$$

7. Find $\int_0^1 (3x + 4)^5 \, dx$

   (a) by first carrying out the indicated multiplication,

   (b) by using the method found in Number 6.

8. Which of the following integrals are the same as $\int_a^b t^3\,dt$?

(a) $\int_a^b y^3\,dy$ (c) $\int_a^b A^3\,dA$

(b) $\int_a^b y^3\,dt$ (d) $\int_{a+1}^{b+1} (t-1)^3\,dt$

9. Evaluate the following integrals using a line of symmetry appropriate to the problem. [e.g., $\int_{-3}^{3} x^2\,dx = 2\int_0^3 x^2\,dx = \frac{2}{3}x^3\Big|_0^3 = 18.$]

(a) $\int_{-\pi/6}^{\pi/6} \cos x\,dx$

(b) $\int_{-2}^{2} (1 + 6x^2)dx$

(c) $\int_0^2 (x-1)^2\,dx$

(d) $\int_0^{\pi} \sin x\,dx$

10. Find the area of the region bounded by the x-axis, the given curves $y = f(x)$, and the given vertical lines. (Sketch first.)

(a) $$f : x \to \left\{\begin{array}{ll} -x^3, & x \leq 0 \\ -x^2 + 2, & 0 < x \leq \sqrt{2} \\ x, & \sqrt{2} < x \end{array}\right\}$$ vertical lines at $x = -1$ and $x = 4$

(b) $$f : x \to \left\{\begin{array}{ll} |2x - 3| & \text{if } 0 \leq x \leq 3 \\ \frac{4}{3}(x - \frac{3}{2})^2 & \text{if } x \leq 0 \\ & \text{or } x \geq 3 \end{array}\right\}$$ vertical lines at $x = -\frac{3}{2}$ and at $x = \frac{9}{2}$

In Problems 11-12 deduce part (b) from the solutions to part (a). (Sketch each first.)

11. (a) (i) Find $\int_0^1 (8 - x^2)dx$ (ii) $\int_0^1 x^2\,dx$

(b) Find the area of the region bounded above by $y = 8 - x^2$, below by $y = x^2$, to the left by the vertical line $x = -1$, and to the right by the vertical line $x = 1$.

12. (a) (i) Find $\int_0^2 (8 - x^2)dx$; (ii) $\int_0^2 x^2\,dx$

(b) Find the area of the region bounded by $y = 8 - x^2$ and $y = x^2$.

13. (a) Find the solution of Number 11(b) directly without using part (a) of Number 11.

(b) Find the solution of Number 12(b) directly without using part (a) of Number 12.

14. Find the area bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$. (Sketch first.)

7-4. Properties of Integrals

We have seen that the integral

$$\int_a^b f \qquad \left(\text{or } \int_a^b f(x)\,dx\right)$$

can be interpreted as the area of the region below the graph of f, above the x-axis and between the vertical lines $x = a$ and $x = b$. Certain properties of integrals are immediately suggested by this area interpretation.

Since the area of a region like that shown in Figure 7-4a should be a nonnegative number; we have the result:

(1) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f \geq 0$

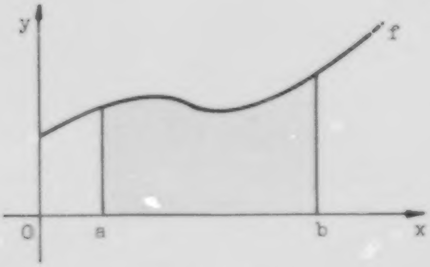


Figure 7-4a

Area under a graph.

Also the area of a region should not exceed the area of any larger region. A useful formulation of this idea is the following:

(2) If $f(x) \leq g(x)$, for $a \leq x \leq b$, then

$$\int_a^b f \leq \int_a^b g.$$

(See Figure 7-4b.)

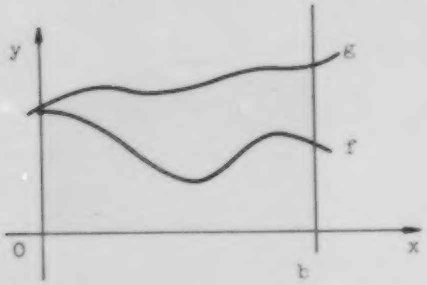


Figure 7-4b

The area under f does not exceed the area under g.

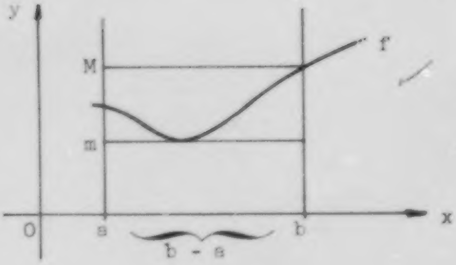


Figure 7-4c

$$m(b - a) \le \int_a^b f \le M(b - a)$$

An application of the inequality in (2) gives bounds for the area in terms of bounds for f. Suppose M is a constant and $f(x) \le M$ for $a \le x \le b$. With $g : x \to M$ we can apply (2) to obtain

$$\int_a^b f \le \int_a^b g = M(b - a).$$

Similar arguments can be applied if $m \le f(x)$ to obtain $m(b - a) \le \int_a^b f$. (See Figure 7-4c.) In summary:

(3) if $m \le f(x) \le M$ for $a \le x \le b$ then

$$m(b - a) \le \int_a^b f \le M(b - a).$$

A line has no width and hence, has zero area. Thus, if we take $b = a$, we should expect the area to be zero, that is,

(4) $$\int_a^a f = 0$$

This is consistent with the result (3) for if we take $b = a$ we obtain

$$0 = m \times 0 \le \int_a^a f \le M \times 0 = 0.$$

If we multiply ordinates by a constant factor then we expect the area to be changed by a corresponding factor. One useful consequence of this:

(5) If $g(x) = \alpha f(x)$, for $a \leq x \leq b$, where α is a positive constant, then

$$\int_a^b g = \alpha \int_a^b f.$$

(See Figure 7-4d.)

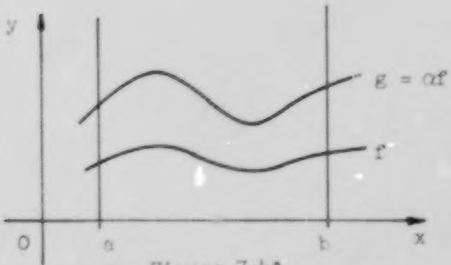


Figure 7-4d

The area under g is α times the area under f.

If one region is the union of two non-overlapping regions we expect the area of the first region to be the sum of the areas of the subregions. This additivity principle has two useful consequences, (6) and (7)..

(6) If c lies between a and b, then

$$\int_a^b f = \int_a^c f + \int_c^b f;$$

that is, if we cut the region under f by a vertical line, then the area is the sum of the two resulting areas. (See Figure 7-4e.)

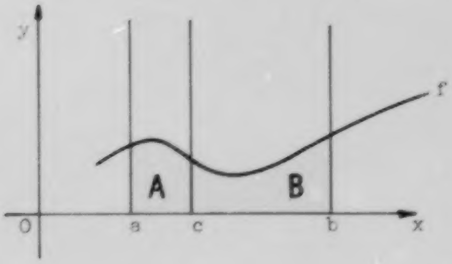


Figure 7-4e

The area of the region under the graph of f between a and b is the sum of the areas of regions A and B.

A second useful formulation of additivity is obtained for the sum of two graphs. The sum $f + g$ is defined as the function whose value at x is $f(x) + g(x)$; that is, the graph of $f + g$ is obtained by adding the ordinates of the graphs of f and g. We have

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g. \tag{7}$$

(See Figure 7-4f).

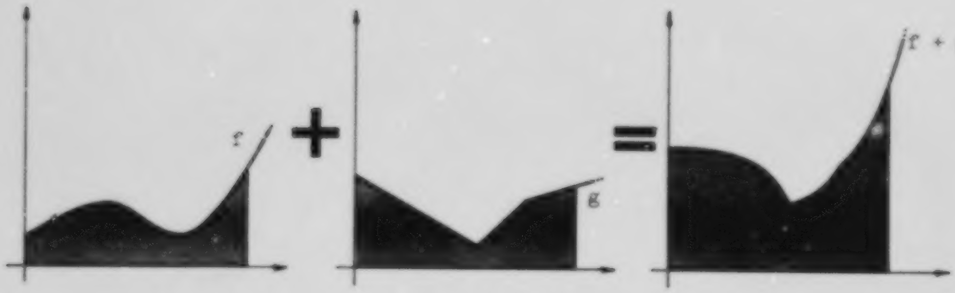


Figure 7-4f

The area of the region under the graph of f plus
the area of the region under the graph of g is
the area of the region under the graph of $f + g$.

Each of these principles can be deduced from the Fundamental Theorem. We prove several of them here, leaving the others as exercises.

Example 7-4a. Prove that

(1) if $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f \geq 0$.

Let $$F'(x) = f(x).$$

Since $$f(x) \geq 0 \text{ for } a \leq x \leq b$$
$$F'(x) \geq 0 \text{ for } a \leq x \leq b$$

and F increases on the interval $[a,b]$. Hence,

$$F(b) \geq F(a).$$

Then $$\int_a^b f = F(b) - F(a) \geq 0.$$

Example 7-4b. Prove that

(5) $$\text{if } g(x) = \alpha f(x) \text{ for } a \leq x \leq b$$

where α is a positive constant, then

$$\int_a^b g = \alpha \int_c^b f.$$

Let $$G'(x) = g(x)$$

and $$F'(x) = f(x).$$

Then $$(\alpha F(x))' = \alpha f(x) = g(x).$$

Since G and αF have the same derivative

$$G(x) = \alpha F(x) + C.$$

Now
$$\begin{aligned}
\int_a^b g &= G(b) - G(a) \\
&= [\alpha F(b) + C] - [\alpha F(a) + C] \\
&= \alpha [F(b) - F(a)] \\
&= \alpha \int_a^b f.
\end{aligned}$$

Note: This proof is equally valid if α is a negative constant.

Exercises 7-4

1. Prove (4), that is,

$$\int_a^a = 0.$$

2. Prove (2) by using the fact that $\int_a^b f \leq \int_a^b g$ is equivalent to $\int_a^b (g - f) \geq 0$ and then using (1).

3. Prove that $D\int_a^x f = f(x)$. Hint: Let $F' = f$ and apply the Fundamental Theorem.

4. Prove (7) by showing that $\int_a^x f + g$ and $\int_a^x f + \int_a^x g$ have the same derivative and that they are equal at $x = a$. Hint: Use Number 3.

5. Suppose $f : x \to x^2$, $g : x \to 2x + 3$.

   (a) Graph each.

   (b) Show that $f(x) \leq g(x)$ for $0 \leq x \leq 3$.

   (c) Verify $\int_0^3 f \leq \int_0^3 g$.

6. Over the indicated interval for the following functions: graph the function; find the maximum (M) value of the function; find the minimum (m) value of the function; and, using these, express with an inequality the lower and upper bounds of the integral expression for the area. [Hint: See Figure 7-2c.]

   (a) $f : x \to x + 1$, $0 \leq x \leq 1$

   (b) $f : x \to x^2 - 2x + 3$, $0 \leq x \leq 3$

7. For $f : x \to 3x - 2$ and $g = \sqrt{2} f$ find $\int_5^{10} f$, $\int_5^{10} g$ and verify that $\int_5^{10} g = \sqrt{2}\int_5^{10} f$.

8. For $f : x \to -2x + 20$ and $g : x \to -2(x - h) + 20$.

   (a) Find a suitable translation such that $f(3) = g(0)$ show that $f(7) = g(4)$. Graph f and g.

   (b) Find $\int_0^3 f$, $\int_0^4 g$, $\int_0^7 f$ and verify that $\int_0^7 f = \int_0^3 f + \int_0^4 g$.

   Thus $\int_0^7 f = \int_0^3 f + \int_3^7 f$.

9. For $f : x \to 3x + 5$, $g : x \to x$ and $h : x \to 1$ verify that

   $$\int_a^b f = 3\int_a^b g + 5\int_a^b h.$$

10. Find each of the following integrals, after first graphing the given function over the interval.

    (a) $\int_1^3 (x^2 + x)dx$

    (b) $\int_1^4 (x^2 - 4x + 5)dx$

    (c) $\int_1^3 (-x^2 + 2x + 3)dx$

    (d) $\int_2^4 (\frac{1}{4} x^2 + \frac{1}{2} x - 1)dx$

11. Suppose $f : x \to px^2 + qx + r$ where p, q and r are nonnegative constants.

    (a) Put $F : x \to \frac{p}{3} x^3 + \frac{q}{2} x^2 + rx$ and show that $F' = f$.

    (b) Show that if $0 \leq a \leq b$ then

    $$\int_a^b f = F(b) - F(a)$$

    (Hint: $\int_a^b f = \int_0^b f - \int_0^a f$)

12. In Exercises 7-1, Number 2 it was shown that for $f : x \to x^3$

$$\int_0^x f = \frac{1}{4} x^4 \quad \text{for} \quad x \geq 0.$$

Suppose $g : x \to px^3 + qx^2 + rx + s$, where p, q, r and s are nonnegative constants. Suppose also that

$$G : x \to \frac{p}{4} x^4 + \frac{q}{3} x^3 + \frac{r}{2} x^2 + sx.$$

(a) Show that $G' = g$.

(b) Show that if $0 \leq a \leq b$ then $\int_a^b g = G(b) - G(a)$.

13. In Number 11 put $G(x) = F(x) + 1000$ and show that $\int_a^b f = G(b) - G(a)$.

14. Find $\int_0^5 |x - 2| dx$. (Hint: A graph is, of course, helpful.)

15. Find $\int_{-10}^{-3} x^2 \, dx$.

16. Find the area and graph of the region bounded by $y = 2(x - 5)^2 - 2$ and $y = 0$. (Hint: Translate and graph the area into the first quadrant.)

17. Find the area of the region bounded by $y = -(x + 1)^2 + 1$ and $y = x$.

7-5. Signed Area

Until now we have discussed the integral $\int_a^b f$ or $\int_a^b f(x)dx$ only in cases for which $a \le b$ and the interval from a to b could be subdivided so that in each subinterval the function f was nonnegative, always increasing (or always decreasing) and its graph had no gaps. We now extend our discussion to include situations for which $a > b$ or for which the graph of f may contain portions below the x-axis, preserving, if possible, the result

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{if} \quad F' = f.$$

This can be accomplished by suitably interpreting $\int_a^b f(x)dx$ as signed area.

First consider the case for which f is nonpositive on the interval $a \le x \le b$, and $F' = f$. In this case $-f$ is nonnegative and has antiderivative $-F$, so that

$$(1) \qquad \int_a^b -f(x)dx = -F(x) \Big|_a^b = -F(b) + F(a).$$

This can be interpreted as the area of the shaded region of Figure 7-5b. Note that this is the same as the area of the shaded region of Figure 7-5a.

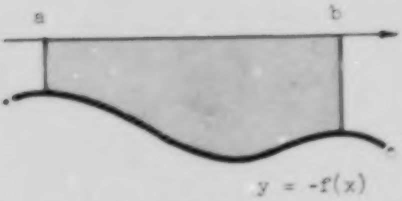


Figure 7-5a

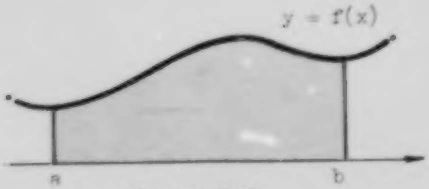


Figure 7-5b

If the Fundamental Theorem is to hold we should have

$$\int_a^b f(x)dx = F(b) - F(a).$$

Referring to (1), we see that this requires that

$$\int_a^b f(x)dx = -\int_a^b [-f(x)]dx;$$

that is $\int_a^b f(x)dx$ must be defined as the negative of the area of the shaded region of Figure 7-5a.

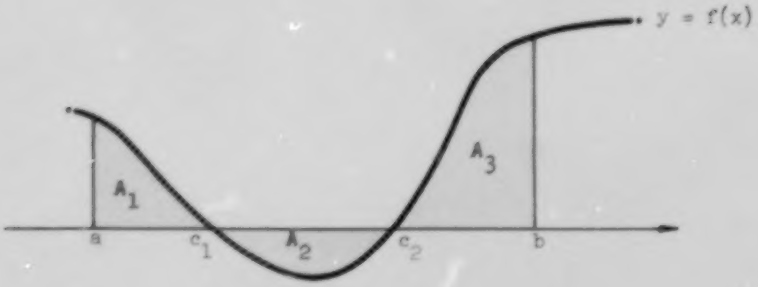


Figure 7-5c

Now suppose the graph of f looks like that shown in Figure 7-5c and that F is an antiderivative of f. We have

$$\text{area of } A_1 = \int_a^{c_1} f(x)dx = F(c_1) - F(a)$$

$$\text{area of } A_2 = \int_{c_1}^{c_2} -f(x)dx = F(c_1) - F(c_2)$$

$$\text{area of } A_3 = \int_{c_2}^{b} f(x)dx = F(b) - F(c_2).$$

Now note that

$$\begin{aligned} F(b) - F(a) &= F(b) - F(c_1) + F(c_1) - F(c_2) + F(c_2) - F(a) \\ &= [F(c_1) - F(a)] - [F(c_1) - F(c_2)] + [F(b) - F(c_2)] \\ &= (\text{area of } A_1) - (\text{area of } A_2) + (\text{area of } A_3). \end{aligned}$$

In other words, if we wish

$$\int_a^b f \text{ to be } F(b) - F(a)$$

then we must have

$$\int_a^b f = (\text{area of } A_1) - (\text{area of } A_2) + (\text{area of } A_3).$$

In summary, if $a \leq b$, $F' = f$ and if we define $\int_a^b f$ by

$$\int_a^b f = F(b) - F(a), \tag{2}$$

then $\int_a^b f$ will be the total area of the regions bounded by the graph of f which lie above the interval minus the total area of the regions bounded by the graph of f which lie below the interval. This is called the signed area determined by f on the interval from a to b.

It is also common practice to remove the restriction that $a \leq b$, by defining

$$\int_a^b f = -\int_b^a f \quad \text{if} \quad b < a.$$

The fundamental relation (2) will still hold, for if $b < a$ and $F' = f$ then

$$\begin{aligned}\int_a^b f = -\int_b^a f &= -[F(a) - F(b)]\\ &= F(b) - F(a).\end{aligned}$$

The properties of the symbol $\int_a^b f$ discussed in Section 7-4 also hold for signed area:

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g; \tag{3}$$

$$\int_a^b (\alpha f) = \alpha\int_a^b f, \quad \text{where } \alpha \text{ is } \underline{\text{any}} \text{ real number;} \tag{4}$$

$$\int_a^b f = \int_a^c f + \int_c^b f, \quad \text{where } a, b, c \text{ are } \underline{\text{any}} \text{ real numbers.} \tag{5}$$

Notice, in fact, that (4) now holds without the restriction that α be non-negative and (5) doesn't require that $a \leq c \leq b$.

Of course, if $a \leq b$ and $f(x) \geq 0$ for $a \leq x \leq b$ then

$$\int_a^b f(x)dx \geq 0.$$

One consequence of this is the fact that

$$(6) \qquad \int_a^b f(x)dx \le \int_a^b g(x)dx \quad \text{if} \quad a \le x \le b \quad \text{and} \quad f(x) \le g(x).$$

For we then have $g(x) - f(x) \ge 0$, so that

$$\int_a^b (g(x) - f(x))dx \ge 0.$$

Adding $\int_a^b f(x)dx$ to both sides, we obtain (6).

Example 7-5a. Find $\int_{-\pi}^{\pi} \sin x \, dx$.

This integral can be interpreted as the signed area of the total shaded region shown in Figure 7-5d. Since the regions above and below the x-axis are the same, we should expect that the signed area is 0. The defining relation (2) should corroborate our expectation. In this case

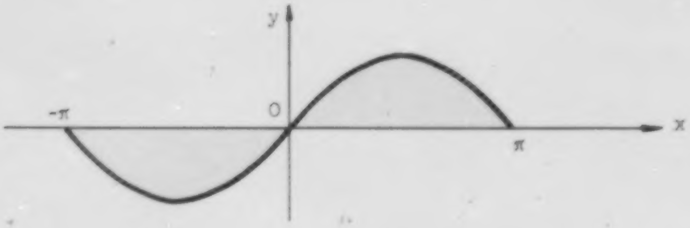


Figure 7-5d

$y = \sin x$

$$F : x \to -\cos x$$

is an antiderivative of $x \to \sin x$, so (2) gives

$$\int_{-\pi}^{\pi} \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi} = (-\cos \pi) - (-\cos(-\pi))$$

$$= (-(-1)) - (-(-1)) = 0.$$

Example 7-5b. Sketch the graph of $f : x \to 1 - x^2$ for $-2 \le x \le 3$.

Find $A = \int_{-2}^{-1} [-f(x)]dx$, $B = \int_{-1}^{1} f(x)dx$, and $C = \int_{1}^{3} [-f(x)]dx$.

Use the fundamental relation (2) to show that

$$\int_{-2}^{3} (1 - x^2)dx = -A + B - C.$$

The desired graph is shown in Figure 7-5e.

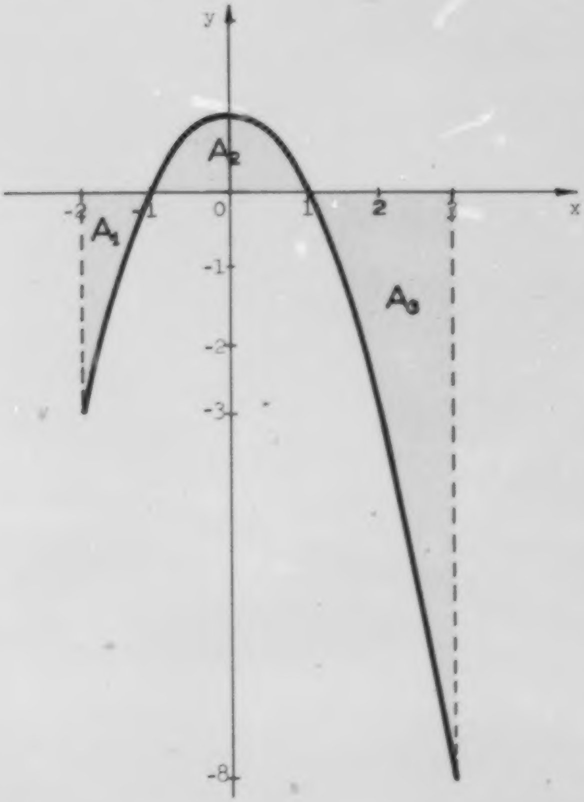


Figure 7-5e

$y = 1 - x^2$

The function $F : x \to x - \frac{1}{3}x^3$ is an antiderivative for f (as easily checked by showing that $F' = f$). We have

$$\int_{-2}^{-1} [-f(x)]dx = \int_{-2}^{-1} (x^2 - 1)dx = \frac{x^3}{3} - x \Big|_{-2}^{-1} = \frac{4}{3} = \text{area of } A_1;$$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (1 - x^2)dx = x - \frac{x^3}{3} \Big|_{-1}^{1} = \frac{4}{3} = \text{area of } A_2;$$

$$\int_{1}^{3} [-f(x)]dx = \int_{1}^{3} (x^2 - 1)dx = \frac{x^3}{3} - x \Big|_{1}^{3} = \frac{20}{3} = \text{area of } A_3;$$

The fundamental relation (2) gives

$$\int_{-2}^{3} f(x)dx = F(3) - F(-2) = x - \frac{x^3}{3} \Big|_{-2}^{3} = -\frac{20}{3},$$

which is the same as

$$-(\text{area of } A_1) + (\text{area of } A_2) - (\text{area of } A_3) = -\frac{4}{3} + \frac{4}{3} - \frac{20}{3} = -\frac{20}{3}.$$

Example 7-5c. Find $\int_{1}^{0} x^2\,dx$ and $-\int_{0}^{1} x^2\,dx$.

We have

$$\int_{1}^{0} x^2\,dx = -\int_{0}^{1} x^2\,dx = -\frac{x^3}{3}\Big|_{0}^{1} = -\frac{1}{3}.$$

Example 7-5d. Find the area of the region enclosed by the graphs of the two functions

$$f : x \to x^2 - 6x + 7 \quad \text{and} \quad g : x \to -x^2 + 7x - 11.$$

A sketch of the region whose area is sought in given in Figure 7-5f.

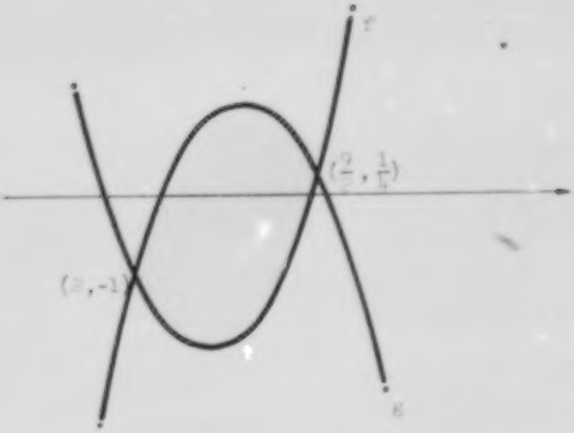

f
$(\frac{9}{2}, \frac{1}{4})$
(2,-1)
g

Figure 7-5f

We shall show that the desired area is given by

$$\int_2^{9/2} (g(x) - f(x))dx.$$

First we note that

(7) $\int_2^{9/2} g(x)dx = -(\text{area of } A_1) + (\text{area of } A_2) + (\text{area of } A_3)$,

where A_1, A_2 and A_3 are the regions indicated in Figure 7-5g.

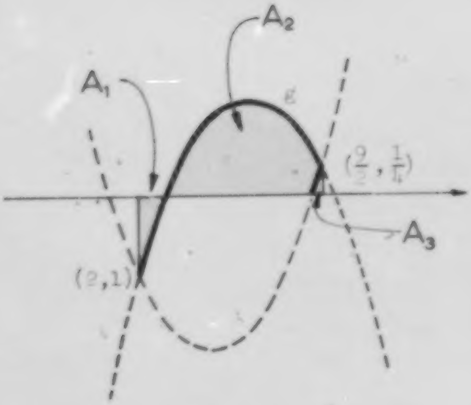

A_2
A_1
g
$(\frac{9}{2}, \frac{1}{4})$
A_3
(2,1)

Figure 7-5g

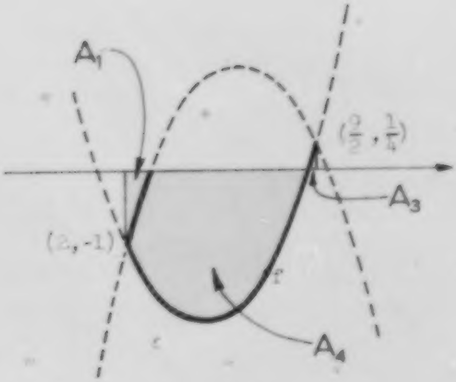

A_1
$(\frac{9}{2}, \frac{1}{4})$
A_3
(2,-1)
f
A_4

Figure 7-5h

Then we observe that

$$(8)\qquad \int_2^{9/2} f(x)dx = -(\text{area of } A_1) - (\text{area of } A_4) + (\text{area of } A_3),$$

where region A_4 is indicated in Figure 7-5h.

Subtracting (8) from (7), we obtain

$$\int_2^{9/2} g(x)dx - \int_2^{9/2} f(x)dx = (\text{area of } A_2) + (\text{area of } A_4),$$

which is the area we seek. Since

$$\int_2^{9/2} g(x)dx - \int_2^{9/2} f(x)dx = \int_2^{9/2} (g(x) - f(x))dx,$$

we establish that $\int_2^{9/2} (g(x) - f(x))dx$ determines the area of the region between the graphs of g and f. A simple calculation now gives

$$\int_2^{9/2} (g(x) - f(x))dx = \int_2^{9/2} (-2x^2 + 13x - 18)dx$$

$$= -\frac{2}{3}x^3 + \frac{13}{2}x^2 - 18x \Big|_2^{9/2} = \frac{125}{24}.$$

Exercises 7-5

1. (a) Sketch the graph of the function

$$f : x \to x^2 - 1, \quad -1 < x < 2.$$

(b) Evaluate $\int_0^2 (x^2 - 1)dx$.

(c) Find the area of the region bounded by the x-axis and the graph of the function, $x \to x^2 - 1$, between the vertical lines at $x = 0$ and $x = 2$.

2. (a) Sketch the graph of the function

$$f : x \to x^3, \quad |x| \leq 1.$$

(b) Evaluate $\int_{-1}^1 x^3\, dx$.

(c) Find the area of the region between the graph of the function, $x \to x^3$ and the x-axis, where $|x| \leq 1$.

(d) Find b $(b > 0)$, if $\int_0^b x^3\, dx = \frac{1}{2}\int_0^2 x^3\, dx$. Sketch.

3. (a) Evaluate $\int_{-1}^1 x\, dx$.

(b) Evaluate $\int_{-1}^1 |x| dx$.

(c) Sketch and then find the area bounded by the x-axis, $|x| = 1$ and $y = x$.

(d) Sketch and then find the area bounded by the x-axis, $|x| = 1$ and $y = |x|$.

4. Sketch and then find the area of the region bounded by the coordinate axes and the curve

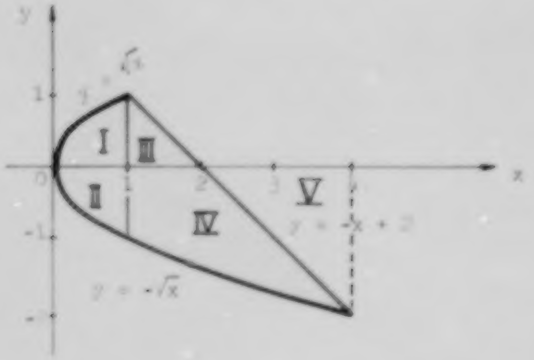
$$\sqrt{x} + \sqrt{y} = 1.$$

Can you identify the curve?

5. Sketch and then find the area of the region bounded by $x = 4$ and $2x = y^2$.

6. Sketch and then find the area of the region bounded by $y = x^3$, $y = -2x^2$ between the vertical lines $x = 0$ and $x = 1$.

7.



Find the area of the region bounded by $y^2 = x$ and $x + y = 2$, indicated in the figure above.

(a) For the first method divide the required region into smaller regions which can be evaluated as follows:

$$A = \int_0^1 \sqrt{x}\,dx + \int_0^1 -(-\sqrt{x})dx + \int_1^2 (-x+2)dx + \left[\int_1^4 [-(-\sqrt{x})]dx - \int_2^4 [-(-x+2)]dx\right]$$

$$A = \quad A_I \quad + \quad A_{II} \quad + \quad A_{III} \quad + \quad [\quad A_{IV} \quad - \quad A_V \quad]$$

Identify this smaller region with their respective integrals.

4 (b) Second, try dividing the required region into different smaller regions which are evaluated as follows:

$$A = \int_0^1 \sqrt{x}\,dx + \int_1^2 (-x + 2)dx + \left[\int_0^4 [-(-\sqrt{x})]dx - \int_2^4 [-(-x + 2)]dx\right]$$

$$A = \quad A_X \quad + \quad A_Y \quad + \quad [\quad A_Z \quad - \quad A_W \quad]$$

Identify the smaller regions with their respective integrals.

(c) Show that the expressions of area in part (a) and part (b) may be simplified to the following statement.

$$A = 2\int_0^1 \sqrt{x}\,dx + \int_1^4 [(-x + 2) + \sqrt{x}]dx$$

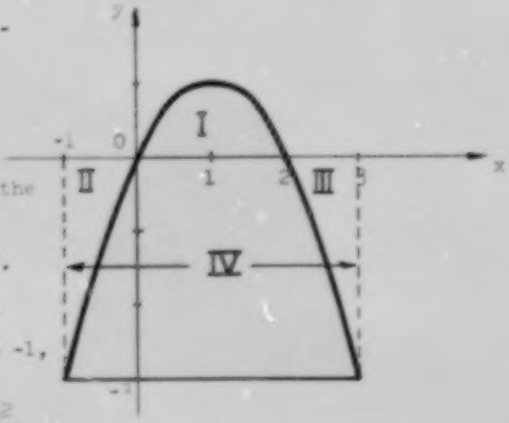
Can you point out the relationship of this expression for the area and the figure representing the area? Could you have arrived at this expression without going through the smaller sub-regions of parts (a) and (b)?

(d) From the expression for the area in part (c) find the area of the region indicated in the figure.

8. (a) Express an integral representing the area of each of the following regions: (DO NOT EVALUATE.)

   (i) Region I: bounded by the x-axis and $y = 2x - x^2$.

   (ii) Region II: bounded by $y = 0$, $x = -1$, and $y = 2x - x^2$.



   (iii) Region III: bounded by $y = 0$, $x = 3$, and $y = 2x - x^2$.

   (iv) Region IV: bounded by $y = 0$, $y = -3$, $x = -1$, and $x = 3$.

   (b) Combine the integrals of part (a) and show that the area of the region bounded by $y = 2x - x^2$ and $y = -3$ can be expressed by the integral,

$$A = \int_{-1}^{3} (2x - x^2 + 3)\,dx.$$

   (c) Find the area of the region described in part (b).

9. (a) Find the area bounded only by the graphs of the functions

$$\begin{cases} f : x \to \cos x \\ f : x \to -\sin x \end{cases}$$

   if x is restricted to the closed interval $-\pi \leq x \leq \pi$. Sketch the curves in this interval.

   (b) (i) Evaluate $\int_{-\pi/4}^{3\pi/4} \cos x \, dx$.

   (ii) Evaluate $\int_{-\pi/4}^{3\pi/4} (-\sin x)\,dx$

   (iii) Evaluate $\int_{-\pi/4}^{3\pi/4} (\cos x - \sin x)\,dx$

   (iv) Interpret parts (i), (ii), and (iii) geometrically.

10. (a) Use a geometric argument to find

$$\int_{-a}^{a} f \quad \text{if } f \text{ is an } \underline{\text{odd}} \text{ function (i.e., } f(-x) = -f(x)).$$

(b) Show that $\int_{-a}^{a} f = 2\int_{0}^{a} f$ if f is an even function (i.e., $f(-x) = f(x)$).

(c) Evaluate $\int_{-5}^{5} (x^3 - 3x)\sin x^2 \, dx$.

11. Show that if $F' = f$, $G' = g$, and $f(x) \le g(x)$ for $a \le x \le b$ then

$$F(b) - F(a) \le G(b) - G(a).$$

12. Verify (5). (Hint: $\int_{a}^{b} f = F(b) - F(a)$.)

13. Suppose $F(x) = \int_{x}^{1} f$ where $f : x \to e^x$.

(a) What is $F(1)$?

(b) Find an expression for $F(x)$.

(c) Use part (b) to find $F'(x)$.

(d) In general, suppose $G(x) = \int_{x}^{b} g$. Can you find $G'(x)$?

14. (a) Find the area bounded by the x-axis and the curve $y = x^2 - x^3$. Sketch.

(b) Find the area bounded by the y-axis and the curve $x = y^2 - y^3$. Sketch. (Hint: Note analogy to part (a).)

7-6. Integration Formulas

We have seen that the integral $\int_a^b f(x)dx$ can be evaluated, if we can find a function F such that $F' = f$, for then we have

$$\int_a^b f(x)dx = F(b) - F(a).$$

In general we find antiderivatives by one or a combination of methods. A method may consist of recalling a differentiation formula, judicious guessing, or using tables of antiderivatives. In this section we review some of the basic formulas used previously, give some additional formulas and discuss the use of tables. Techniques for extending the scope of our formulas will be discussed in Chapter 9, where we also discuss methods for obtaining approximate values for integrals. Other integration methods are discussed in the appendices.

The common notation for an antiderivative of f is

$$\int f(x)dx,$$

which is also called the indefinite integral of f. This symbol is quite similar to

$$\int_a^b f(x)dx,$$

the integral of f from a to b. The symbol

$$\int f(x)dx$$

defines a function, namely, a function whose derivative is f. The second symbol

$$\int_a^b f(x)dx$$

represents a number, which can be interpreted as the signed area determined by f between a and b.

Integration formulas are obtained by reversing the differentiation process, for

$$\int f(x)dx = F(x) \quad \text{means that} \quad DF(x) = f(x).$$

For example,

$$\int x^2\, dx = \frac{x^3}{3} \quad \text{since} \quad D\,\frac{x^3}{3} = x^2.$$

Of course, if C is any constant, we have

$$D(\frac{x^3}{3} + C) = x^2;$$

more precisely we have

$$\int x^2\, dx = \frac{x^3}{3} + C.$$

In fact, we know from the Constant Difference Theorem (Theorem 7-3b) that all antiderivatives of $x \to x^2$ have the form

$$x \to \frac{x^3}{3} + C, \quad \text{where } C \text{ is a constant.}$$

In some books this fact is stressed by writing

$$\int f(x)dx = F(x) + C,$$

where C is a constant and $DF(x) = f(x)$. For convenience we follow the simple practice of ignoring this constant C in our formulas, each integration formula giving only one function whose derivative is f. Others are obtained by adding constants to our antiderivatives.

The Power Formula

Recall that if a is any real number then

$$Dx^a = ax^{a-1}.$$

If $a \neq 0$, we can write

$$D(\frac{1}{a}\, x^a) = x^{a-1},$$

so that $x \to \frac{1}{a} x^a$ is a function whose derivative is $x \to x^{a-1}$. This tells us that

$$\int x^{a-1}\,dx = \frac{1}{a} x^a, \quad \text{if} \quad a \neq 0.$$

For convenience we replace a by $p + 1$, where p is any real number except $p \neq -1$, to obtain the formula

$$\int x^p\,dx = \frac{x^{p+1}}{p+1}, \quad p \neq -1.$$

In other words, an antiderivative of a power function $x \to x^p$, $p \neq -1$, is obtained by raising the exponent by 1 and dividing by the new exponent.

Suppose $p = -1$, then our function is $x \to \frac{1}{x}$. In Section 6-6 we obtained the formula

$$D \log_e x = \frac{1}{x}, \quad x > 0.$$

This gives the integration formula

$$\int \frac{1}{x}\,dx = \log_e x, \quad x > 0.$$

Circular and Exponential Functions

From the formulas

$$D \sin x = \cos x; \quad D \cos x = -\sin x,$$

we obtain the integration formulas

$$\int \cos x\,dx = \sin x; \quad \int \sin x\,dx = -\cos x.$$

Since $De^x = e^x$, we have the formula

$$\int e^x\,dx = e^x.$$

It is a simple matter to extend these formulas to the case when x is replaced by the linear expression $cx + d$. For example, we know that

$$D \sin (cx + d) = c \cos (cx + d)$$

so that

$$\int c \cos (cx + d)dx = \sin (cx + d).$$

If $c \neq 0$, we can write

$$\int \cos (cx + d)dx = \frac{1}{c} \sin (cx + d).$$

Analogous differentiation formulas were discussed in Volume One for polynomial, exponential and logarithmic functions. In Chapter 9 we shall discuss the formulas resulting from nonlinear substitutions. Here we state the general result for linear replacements:

$$\boxed{\begin{array}{ll} \text{if} & \int f(x)dx = F(x) \quad \text{and} \quad c \neq 0, \\ \text{then} & \int f(cx + d)dx = \frac{1}{c} F(cx + d). \end{array}}$$

For easy reference we summarize current results in Table 7-6.

Table 7-6

Some Integration Formulas

(1) $\int x^a \, dx = \frac{x^{a+1}}{a + 1}, \quad a \neq -1$

(2) $\int \frac{1}{x} \, dx = \log_e x, \quad x > 0$

(3) $\int \cos x \, dx = \sin x$

(4) $\int \sin x \, dx = -\cos x$

(5) $\int e^x \, dx = e^x$

(6) $\int f(cx + d)dx = \frac{1}{c} F(cx + d)$

Example 7-6a. Find $\int_1^{3/2} \frac{1}{x^2}\,dx$.

The power formula (1), with $a = -2$, gives

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} = -\frac{1}{x}\ ;$$

so that

$$\int_1^{3/2} \frac{1}{x^2}\,dx = -\frac{1}{x}\Bigg|_1^{3/2} = \left(-\frac{1}{\frac{3}{2}}\right) - \left(-\frac{1}{1}\right) = \frac{1}{3}\ .$$

Example 7-6b. Find $\int_2^4 \sqrt{x}\,dx$.

The power formula (1), with $a = \frac{1}{2}$, gives

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2x^{3/2}}{3}\ ;$$

so that

$$\int \sqrt{x}\,dx = \frac{2x^{3/2}}{3}\Bigg|_2^4 = \frac{2}{3}(4^{3/2} - 2^{3/2})$$

$$= \frac{16 - 4\sqrt{2}}{3}\ .$$

Example 7-6c. Find

$$\int_0^{\pi} (\sin x - 3\cos 2x)dx.$$

We have, from (4) and (3),

$$\int \sin x\,dx = -\cos x \quad \text{and} \quad \int \cos x\,dx = \sin x.$$

Replacing x by $2x$ in the latter and using (6), we have

$$\int \cos 2x\,dx = \frac{1}{2}\sin 2x.$$

Therefore, we conclude

$$\int_0^\pi (\sin x - 3 \cos 2x)dx = \int_0^\pi \sin x \, dx - 3 \int_0^\pi \cos 2x \, dx$$

$$= -\cos x \Big|_0^\pi - \frac{3}{2} \sin 2x \Big|_0^\pi$$

$$= -[\cos \pi - \cos 0] - [\frac{3}{2} \sin 2\pi - \frac{3}{2} \sin 0]$$

$$= -[-1 - 1] - [\frac{3}{2} \cdot 0 - \frac{3}{2} \cdot 0] = 2.$$

Example 7-6d. Find $\int_{-10}^{-1} 2 e^x \, dx$.

We use (5) to obtain

$$\int_{-10}^{-1} 2 e^x \, dx = 2 \int_{-10}^{-1} e^x \, dx = 2 e^x \Big|_{-10}^{-1}$$

$$= 2e^{-1} - 2e^{-10}.$$

Example 7-6e. Find $\int_0^1 2^x \, dx$.

We first convert to base e:

$$2^x = e^{cx}, \text{ where } c = \log_e 2.$$

Now we use (5) to obtain

$$\int e^x \, dx = e^x.$$

We replace x by cx, so that (6) gives

$$\int e^{cx} \, dx = \frac{1}{c} e^{cx},$$

where $c = \log_e 2$. Converting to base 2, we have

$$\int 2^x \, dx = (\frac{1}{\log_e 2}) 2^x;$$

so that

$$\int_0^1 2^x \, dx = \left(\frac{1}{\log_e 2}\right) 2^x \Big|_0^1 = \frac{2^1 - 2^0}{\log_e 2} = \frac{1}{\log_e 2} .$$

Example 7-6f. Find $\int_{-1}^{0} (x + 1)^3 dx$.

We can evaluate this integral in two ways. First we expand to obtain

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1,$$

so that

$$\int_{-1}^{0} (x + 1)^3 dx = \int_{-1}^{0} (x^3 + 3x^2 + 3x + 1)dx.$$

We apply the power formula (1) to each term to obtain

$$\int_{1}^{0} (x + 1)^3 dx = \left(\frac{x^4}{4} + x^3 + \frac{3x^2}{2} + x\right) \Big|_{-1}^{0} = [0] - [\tfrac{1}{4} - 1 + \tfrac{3}{2} - 1] = \tfrac{1}{4} .$$

Alternatively we can recognize that the power formula (1) gives

$$\int x^3 \, dx = \frac{1}{4} x^4 ,$$

and the linear substitution formula (6) gives

$$\int (x + 1)^3 \, dx = \frac{1}{4}(x + 1)^4 .$$

Therefore, we conclude that

$$\begin{aligned} \int_{-1}^{0} (x + 1)^3 \, dx &= \frac{1}{4}(x + 1)^4 \Big|_{-1}^{0} \\ &= \frac{1}{4}(0 + 1)^4 - \frac{1}{4}(-1 + 1)^4 \\ &= \frac{1}{4} . \end{aligned}$$

The second method is certainly quicker.

Example 7-6g. Find $\int_0^1 \sin^2 \pi x \, dx$.

We have not yet obtained a differentiation formula which results in the square of the sine function. We use the fact that

$$\sin^2 \pi x = \frac{1 - \cos 2\pi x}{2} .$$

Thus, we have

$$\begin{aligned} \int_0^1 \sin^2 \pi x \, dx &= \int_0^1 \left(\frac{1}{2} - \frac{\cos 2\pi x}{2}\right) dx \\ &= \frac{1}{2}\int_0^1 1 \, dx - \frac{1}{2}\int_0^1 \cos 2\pi x \, dx. \end{aligned}$$

To evaluate this second integral, we combine the cosine formula (3) with the linear substitution result (6) to obtain

$$\int \cos(2\pi x) dx = \frac{1}{2\pi} \sin(2\pi x).$$

We can write

$$\begin{aligned} \int_0^1 \cos 2\pi x \, dx &= \frac{1}{2\pi} \sin 2\pi x \Big|_0^1 \\ &= \frac{1}{2\pi}(\sin 2\pi - \sin 0) = 0. \end{aligned}$$

Since the second integral is 0, we conclude that

$$\int_0^1 \sin^2 \pi x \, dx = \frac{1}{2}\int_0^1 1 \, dx - 0 = \frac{1}{2} x \Big|_0^1 = \frac{1}{2} .$$

Example 7-6h. Show that the area of the shaded region of Figure 7-6a is twice that of the shaded region of Figure 7-6b.

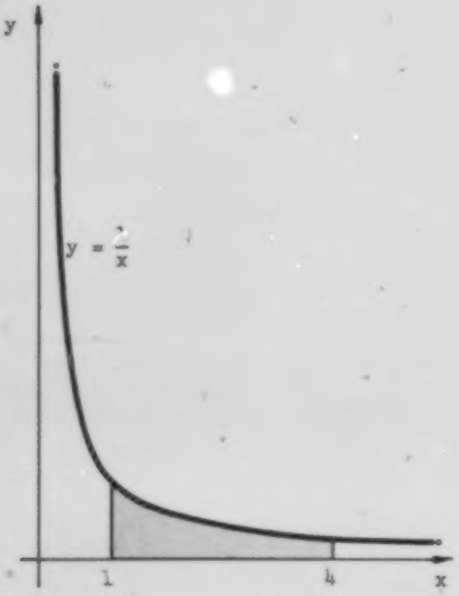


Figure 7-6a

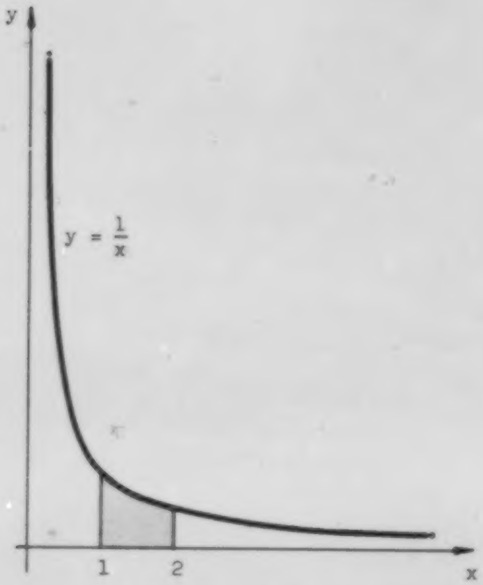


Figure 7-6b

Let $\alpha = \int_1^4 \frac{1}{x}\,dx$ and $\beta = \int_1^2 \frac{1}{x}\,dx$. We wish to show that $\frac{\alpha}{\beta} = 2$.

Formula (2) gives

$$\int \frac{1}{x}\,dx = \log_e x,$$

so that

$$\alpha = \int_1^4 \frac{1}{x}\,dx = \log_e x \Big|_1^4 = \log_e 4 - \log_e 1 = \log_e 4;$$

$$\beta = \int_1^2 \frac{1}{x}\,dx = \log_e x \Big|_1^2 = \log_e 2 - \log_e 1 = \log_e 2.$$

Thus, we conclude that

$$\frac{\alpha}{\beta} = \frac{\log_e 4}{\log_e 2} = \frac{\log_e 2^2}{\log_e 2} = \frac{2 \log_e 2}{\log_e 2} = 2.$$

The Use of Tables

A longer table of integrals is given in a separate booklet (Table 7). As more differentiation methods are developed, we shall see how to construct these tables. The following examples make use of these tables.

Example 7-6i. Find $\int_0^1 xe^x\,dx$.

Formula 16 of the tables gives

$$\int xe^x\,dx = xe^x - e^x,$$

so that

$$\int_0^1 xe^x\,dx = (xe^x - e^x)\Big|_0^1$$

$$= (1e^1 - e^1) - (0e^0 - e^0)$$

$$= 1.$$

Example 7-6j. Find $\int_0^1 xe^{3x}\,dx$.

Formula 16 of the tables gives $\int xe^x\,dx = xe^x - e^x$. We replace x by $3x$ and use (6) to obtain

$$\int 3xe^{3x}\,dx = (3xe^{3x} - e^{3x});$$

so that

$$\int_0^1 x\,e^{3x}\,dx = \frac{1}{3}(3xe^{3x} - e^{3x})\Big|_0^1 = \frac{1}{3}(3e^3 - e^3) - \frac{1}{3}(0e^0 - e^0) = \frac{2}{3}\,e^3 + \frac{1}{3}\,.$$

Example 7-6k. Find $\int_0^1 \log_e(1 + x)dx$.

We use Formula 7 of the booklet tables: $\int \log_e x\,dx = x\log_e x - x$. Replace x by $1 - x$ and use (6) from this chapter to obtain

$$\int_0^1 \log_e(1 + x)dx = [(x + 1)\log_e(x + 1) - (x + 1)]\Big|_0^1$$

$$= (2\log_e 2 - 2) - (1\log_e 1 - 1)$$

$$= 2\log_e 2 - 1.$$

Example 7-6ℓ. Find $\int_{-\pi}^{\pi} \sin^4 x \, dx$.

Formula 28 of the booklet tables gives that

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

With $n = 4$, we have

$$\int \sin^4 x \, dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x \, dx.$$

To find this second integral we can use a trigonometric identity (as in Example 7-6g) or we can use Formula 28 again with $n = 2$ to obtain

$$\begin{aligned} \int \sin^2 x \, dx &= \frac{-\sin x \cos x}{2} + \frac{1}{2} \int 1 \, dx \\ &= \frac{-\sin x \cos x}{2} + \frac{1}{2} x. \end{aligned}$$

Therefore, we have

$$\int_{-\pi}^{\pi} \sin^4 x \, dx = \left(\frac{-\sin^3 x \cos x}{4} - \frac{3 \sin x \cos x}{8} + \frac{3x}{8} \right) \Bigg|_{-\pi}^{\pi}$$

Since $\sin \pi = \sin(-\pi) = 0$, this becomes

$$\frac{3x}{8} \Bigg|_{-\pi}^{\pi} = \frac{3}{8}(\pi - (-\pi)) = \frac{3\pi}{4}.$$

Example 7-6m. Find $\int_{0}^{10} e^{-x^2} \, dx$.

The tables give no formula for $\int e^{-x^2} \, dx$. There is a good reason for this: it is known that there is no elementary function whose derivative is $x \to e^{-x^2}$. Our integral, therefore, can't be found by using the Fundamental Theorem of Calculus and we must resort to some approximation method in order to estimate this integral. We shall have more to say about this in Section 9-4.

Exercises 7-6

For problems 1-15 find the following indefinite integrals.

1. $\int (x^2 + 1)dx$

2. $\int (\frac{1}{x^2} + x + x^4)dx$

3. $\int 8\sqrt{x}\, dx$

4. $\int (x^2 - \sqrt{x})dx$

5. $\int (\frac{1 - x}{x})dx, \quad (x > 0)$ [Hint: Write as 2 fractions.]

6. $\int \sin 3x\, dx$

7. $\int \cos(2x - 5)dx$

8. $\int (-\sin 2x)dx$

9. $\int [-\cos(3x - 1)]dx$

10. $\int \frac{4}{3} \cos 3x\, dx$

11. $\int 2 \sin x \cos x\, dx$ [Hint: Use trigonometric identity.]

12. $\int (3 \sin 2x - 6 \cos 3x)dx$

13. $\int e^{2x}\, dx$

14. $\int e^{x/3}\, dx$

15. $\int (e^x + e^{-x})^2 dx$ [Hint: Remove parenthesis.]

For problems 16-25 find the following indefinite integrals, (using tables when necessary).

16. $\int x^2 e^x \, dx$

17. $\int x^3 e^x \, dx$

18. $\int x^4 e^x \, dx$

19. $\int x^2 \log_e x \, dx$

20. $\int x^3 \log_e x \, dx$

21. $\int x^4 \log_e x \, dx$

22. $\int x^2 \sin x \, dx$

23. $\int x^3 \sin x \, dx$

24. $\int e^{3x} \sin 4x \, dx$

25. $\int e^{x/2} \cos \frac{3x}{2} \, dx$

For problems 26-31, sketch a graph of the relevant region and find the value of the indicated integral.

26. $\int_0^{\pi} (x + \sin x)dx$

27. $\int_0^{2\pi} (x + \sin x)dx$

28. $\int_{-1}^{+1} \frac{e^x + e^{-x}}{2} \, dx$

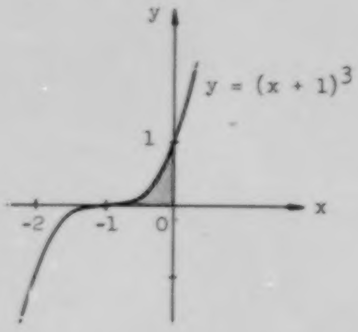
29. $\int_{-1}^{+1} \frac{e^x - e^{-x}}{2} \, dx$

30. $\int_0^{2\pi} x \sin x \, dx$

31. $\int_{1/e^2}^{e^2} \frac{\log_e x}{\sqrt{x}} \, dx$

For problems 32-33, the following instructions are to be followed (linear substitution: translation). In this section we were given an area represented by:

$$A = \int_{-1}^{0} (x + 1)^3 \, dx.$$



By replacement of $x + 1$ by x, (i.e., x by $x - 1$), and by appropriate change of limits, we find an equivalent expression for the area. After the linear substitution, we have

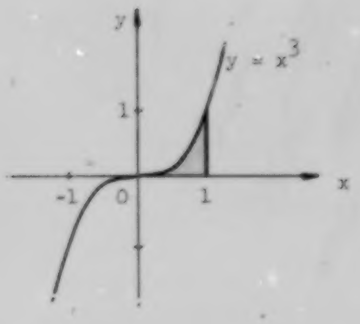
$$A_{L.S.} = \int_0^1 x^3 \, dx \qquad \text{and}$$

evaluating the two equivalent forms of the area:

$$A = \int_{-1}^{0} (x+1)^3 dx = \frac{1}{4}(x+1)^4 \Big|_{-1}^{0} = \frac{1}{4},$$

$$A_{L.S.} = \int_{0}^{1} x^3\, dx = \frac{1}{4}x^4 \Big|_{0}^{1} = \frac{1}{4},$$

we see that they are, indeed, the same.



In the following two problems, follow the format above: Sketch the area defined by the integral, make an appropriate linear substitution, sketch the equivalent area, and evaluate each.

32. $A = \int_{3}^{4} \frac{1}{(x-2)^2} dx$

33. $A = \int_{1}^{2} x(x-1)^3\, dx$

For problem 34-35, follow the instructions of problems 32 and 33, except in this case the linear substitution is a scale change instead of a translation. Draw two graphs as before.

34. $A = \int_{0}^{\pi/2} \sin 2x\, dx$

35. $A = \int_{1}^{4} \sqrt{3x}\, dx$

36. (a) Show that if $x < 0$ then

$$D \log_e(-x) = \frac{1}{x}.$$

(Hint: Sketch $f : x \to \log_e(x)$, $x > 0$ and $g : x \to \log_e(-x)$, $x < 0$.)

(b) Use part (a) to find

$$\int_{-3}^{-1} \frac{1}{x} dx$$

and sketch the area.

37. (a) Can you apply the Fundamental Theorem to find $\int_0^1 \frac{1}{x}\,dx$? If so, do so. If not, state reasons.

(b) Show that

$$\lim_{n \to \infty} \int_{1/n}^{1} \frac{1}{x}\,dx = \infty$$

(c) Use part (b) to discuss what area, if any, you think should be assigned to the region bounded by $y = \frac{1}{x}$, $x \neq 0$, the x and y axes and the vertical line $x = 1$.

(d) What answer seems reasonable to you for $\int_{-1}^{1} \frac{1}{x}\,dx$? With what properties of area is your answer consistent? inconsistent?

Chapter 8

DIFFERENTIATION THEORY AND TECHNIQUE

In Chapters 2, 4, and 6 we showed that the derivative of a polynomial function was also a polynomial function (of one lower degree) and established for certain transcendental functions the formulas:

$$D(\sin x) = \cos x \qquad D(\cos x) = -\sin x$$

$$D(e^x) = e^x \qquad D(\log_e x) = \frac{1}{x} .$$

These are the basic differentiation formulas. Our primary purpose in this chapter is to obtain formulas for differentiating various algebraic combinations of these functions and to use these derivatives to discuss graphs and motion.

The first section of this chapter includes a review of the terminology of derivatives, as well as an introduction to the relationship between continuity and differentiability. Various geometric properties of graphs of continuous functions are illustrated in Section 8-2, where the Intermediate Value Theorem and related theorems on maximum and minimum values of functions over intervals are introduced to establish the connection between derivatives and the shape of the graph of a function. The Mean Value Theorem and applications are discussed in Sections 8-3 and 8-4. As a special case of the Mean Value Theorem, Rolle's Theorem is left to Exercises 8-4, Number 1. Derivatives of sums, multiples and products are discussed in Sections 8-5 and 8-6. Functions which are composites of simpler functions are discussed in Section 8-7 and the important "chain rule" for differentiating such functions is given in Section 8-8. Special cases of the chain rule, which enable us to differentiate powers, reciprocals and quotients are described in Sections 8-9 and 8-10. A general discussion of the "folding" process used in Chapters 5 and 6 to define and differentiate root and logarithmic functions is contained in Section 8-11. These results are applied, in particular, to the inverse trigonometric functions. The final section of this chapter gives a special technique for differentiating functions which are defined implicitly by relations.

8-1. Differentiability

We have often found the derivative of a function. Let us recall the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} .$$

As you know, $f'(x)$ represents the slope of the tangent to the graph of f at the point $(x, f(x))$. A few examples will freshen our memories.

Example 8-1a. If $f : x \to x^2$, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} 2x + h \qquad (h \neq 0) \\ &= 2x. \end{aligned}$$

In different notation, we write

$$Dx^2 = 2x,$$

which we can read "the derivative of x^2 is $2x$."

Example 8-1b.

$$f : x \to \frac{1}{x}$$

$$f'(x) = \lim_{x \to 0} \frac{\frac{1}{x + h} - \frac{1}{x}}{h} .$$

Since
$$\frac{1}{x + h} - \frac{1}{x} = \frac{x - (x + h)}{x(x + h)} ,$$

the numerator can be written as $-\frac{h}{x(x + h)}$; and the difference quotient $(h \neq 0)$ becomes $-\frac{1}{x(x + h)}$. Taking the limit as h approaches zero, we obtain

$$f'(x) = -\frac{1}{x^2} \qquad (x \neq 0)$$

and conclude that
$$D\left(\frac{1}{x}\right) = -\frac{1}{x^2} \qquad (x \neq 0).$$

Example 8-1c. $f : x \to \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} .$$

We transform the difference quotient by multiplying by

$$\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} ,$$

which is 1 in disguise. Then

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \qquad (h \neq 0).$$

If we let h approach zero we obtain

$$f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \qquad (x \neq 0);$$

otherwise stated

$$D\sqrt{x} = \frac{1}{2\sqrt{x}} \qquad (x \neq 0).$$

Example 8-1d. Let $f : x \to \sin x$. Then

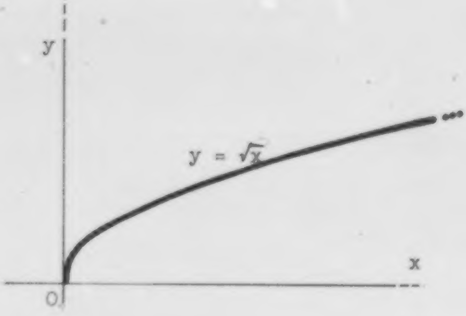
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \to 0} \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right) \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x, \end{aligned}$$

where we use the fact that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

We can write our result as $D \sin x = \cos x$.

Does a function f have a derivative for all values of x for which f is defined? Since $f'(x)$ represents the slope of the tangent at $(x,f(x))$ the question is this: Can there be points on the graph of f at which either there is no tangent at all or a vertical tangent? (We remember that the slope of a vertical line is undefined.) It is not hard to see that the answer is that there can be such points.



For example, the graph of

$$f : x \to \sqrt{x}$$

has a vertical tangent at the origin and therefore f has no derivative when $x = 0$. This appears from the expression for

$$D\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x \neq 0.$$

Since $\frac{1}{2\sqrt{x}}$ fails to exist when $x = 0$, we say that f is differentiable if $x > 0$ but not differentiable at $x = 0$.

A more interesting example is furnished by the absolute value function

$$x \to |x|.$$

Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Its graph consists of two half lines, one of which bisects the first quadrant and the other the second quadrant (Figure 8-1a). Hence, there is a corner at the origin.

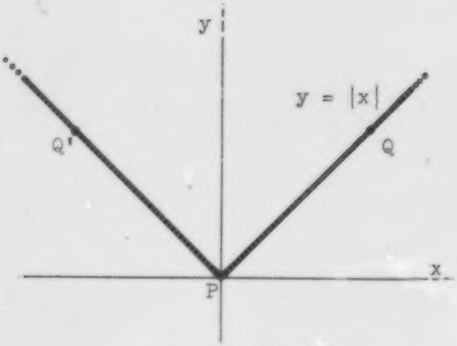


Figure 8-1a

$x \to |x|$

Is there a tangent to the graph at the origin? That is, does $f'(0)$ exist? For $x = 0$, the difference quotient is

$$\frac{|0 + h| - |0|}{h} = \frac{|h|}{h}.$$

Now if $h > 0$, $|h| = h$ and $\frac{|h|}{h} = \frac{h}{h} = 1$.

The slope of $\overleftrightarrow{PQ}$ is 1. If $h < 0$, $|h| = -h$ and $\frac{|h|}{h} = \frac{-h}{h} = -1$. The slope of $\overleftrightarrow{PQ'}$ is -1. The situation is exactly the same whether Q and Q' are close to P or not. If there is to be a tangent at the origin the difference quotient must approach a single limit whether h approaches zero from the right or the left. In this case, therefore, there can be no tangent. Inspection of the graph makes this result reasonable. There is no single line through P which fits the graph closely on both sides. In general, a function will fail to be differentiable at any point where its graph has a "corner."

Consider the function f whose values are given by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

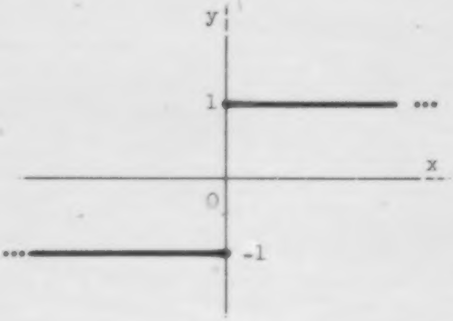


Figure 8-1b

The graph of f is sketched in Figure 8-1b. Note the "jump" at $x = 0$. We say that f is discontinuous (that is, not continuous) at 0, or that f has a discontinuity at $x = 0$. At such a point there cannot be a derivative. To see this, consider what happens if we join $P(0,1)$ to $Q(h,f(h))$ where $h \neq 0$. If $h > 0$, $f(h) = 1$ and Q is $(h,1)$. The slope of $\overleftrightarrow{PQ}$ is zero, whether h is large or small. If $h < 0$, $f(h) = -1$, Q is $(h,-1)$, and therefore, the slope of $\overleftrightarrow{PQ}$ is

$$\frac{-1 - 1}{h} = \frac{2}{-h} .$$

If we take h to be successively -0.1, -0.01, -0.001, ..., we obtain the slopes 20, 200, 2000, ... Clearly, $\frac{2}{-h}$ increases beyond all bounds as h approaches 0 through negative values. Therefore, $f'(0)$ does not exist.

We generalize this result

(1) In order that $f'(a)$ shall exist, it is necessary that f be continuous at a.

How can we show this? So far we have not said exactly what we mean when we say that f is continuous at $x = a$. We have been content to say that there is no "gap" in the graph at $x = a$. We can now be more precise and adopt the following definition.

Definition. A function f is said to be continuous at a if

(1) $f(a)$ is defined

(2) $\lim_{h \to 0} f(a + h) = f(a)$

That is, f must have a value when $x = a$ and moreover this value, $f(a)$, must be approached as h approaches 0, that is, as x approaches a. Let us illustrate.

Example 8-1e. If $f : x \to \frac{1}{x}$, f is not continuous at 0 because $\frac{1}{0}$ is not a number. [$f(0)$ does not exist: there is no such number.] This is enough to establish the conclusion.

However, for good measure we see that $f(0 + h) = \frac{1}{0 + h} = \frac{1}{h}$ does not approach any limit.

Example 8-1f. Let

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

so that the graph is that shown in Figure 8-1b. If our definition is any good, it should tell us that f is not continuous at 0. Let us apply the tests.

(1) Is $f(0)$ defined? Yes, $f(0) = 1$.

(2) Does $f(0 + h)$ approach 1 as h approaches 0? No; in fact, if $h < 0$, $f(0 + h) = f(h) = -1$ and no matter how close to zero h may be chosen, $f(h) = -1$ is no closer to 1 than $1 - (-1) = 2$.

Now that we know what it means to say that f is continuous at a, we are in a position to justify the statement (i), which we repeat for convenience.

(1) In order that $f'(a)$ shall exist, it is necessary that f be continuous at a.

To say that $f'(a)$ exists means that $\frac{f(a + h) - f(a)}{h}$ approaches a limit $f'(a)$ as h approaches zero. Then

$$h \cdot \left(\frac{f(a + h) - f(a)}{h}\right) \text{ approaches } 0 \cdot f'(a) = 0.$$

That is,

$$f(a + h) - f(a) \text{ approaches } 0$$

and hence,

$$f(a + h) \text{ approaches } f(a).$$

But this means that f is continuous at a.

Exercises 8-1

Find any values of x for which the following functions are not differentiable. Give reasons and sketch the graphs.

1. $f : x \to |x - 1|$
2. $f : x \to \dfrac{1}{x + 2}$
3. $f : x \to |\sin x|$
4. $f : x \to \dfrac{1}{x^2}$
5. $f : x \to x^{3/2}$
6. $f : x \to x^{2/3}$
7. Let $f : x \to \sin \frac{1}{x}$, $x > 0$.

   (a) Find $f(\frac{1}{n\pi})$, n a positive integer.

   (b) Find $f\left(\frac{1}{\frac{\pi}{2}}\right)$, $f\left(\frac{1}{\frac{5\pi}{2}}\right)$, $f\left(\frac{1}{\frac{9\pi}{2}}\right)$, ...

   (c) Find $f\left(\frac{1}{\frac{3\pi}{2}}\right)$, $f\left(\frac{1}{\frac{7\pi}{2}}\right)$, $f\left(\frac{1}{\frac{11\pi}{2}}\right)$, ...

   Is there any way to define $f(0)$ so that $\lim_{h \to 0} f(h) = f(0)$?

8-2. Continuous Functions

In the previous section, we showed that a function f must be continuous at any a for which $f'(a)$ exists. Since a polynomial function has a derivative at each value of x, polynomial functions are continuous everywhere. If f is a rational function

$$f : x \rightarrow \frac{p(x)}{q(x)}$$

where p and q are polynomial functions, we know that f is not defined for any value of x for which $q(x) = 0$. At such an x, f is therefore discontinuous. We shall learn that for all other values of x, the derivative $f'(x)$ exists. Therefore, we can conclude that f is continuous when $q(x) \neq 0$.

For example, $f : x \rightarrow \frac{x + 3}{x^2 + 2}$ is continuous everywhere since $x^2 + 2$ is never zero. However, $f : x \rightarrow \frac{x + 3}{x - 1}$ is continuous except at $x = 1$ where it is discontinuous.

The function

$$f : x \rightarrow \frac{\sin x}{x}$$

is discontinuous at 0 since $f(0)$ does not exist. If we define $f(0)$ to be 0 (see Figure 8-2a) the function is still discontinuous since (from Section 4-2)

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \neq f(0).$$

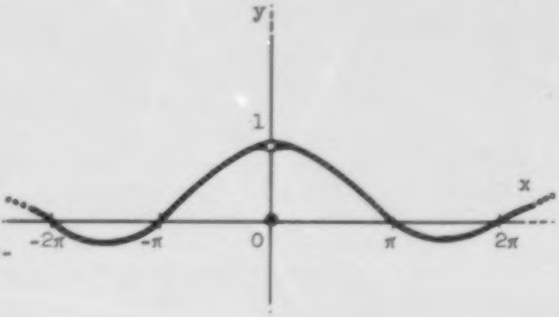


Figure 8-2a

In fact, f is discontinuous at 0 unless we take $f(0) = 1$. In this case, f is continuous everywhere.

There are two important theorems about functions that are continuous at all points on an interval $[a,b]$ which includes both endpoints, that is, an interval, $a \leq x \leq b$.

The Intermediate Value Theorem

> THEOREM 8-2a. If f is continuous on $[a,b]$ with $f(a) = A$ and $f(b) = B$ and if C is a number between A and B then there is at least one number c such that $f(c) = C$.

A special case of this theorem is the Location Theorem in which f is a polynomial function, A and B have opposite signs and C = 0 (see Chapter 1).

We illustrate the theorem for a few examples. We shall not prove the theorem here.

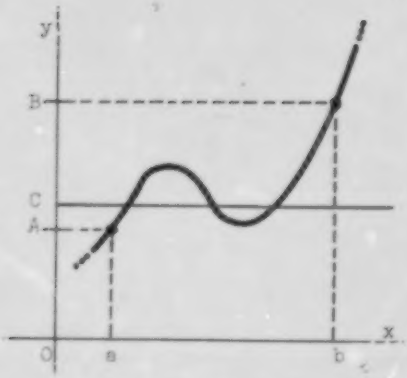


Figure 8-2b

Example 8-2a. Let $f(x) = \frac{x+3}{x^2+2}$ with $a = 0$, $b = 2$, and $C = 1$. Here $f(0) = \frac{3}{2} = A$ and $f(2) = \frac{5}{6} = B$. Since C is between $\frac{3}{2}$ and $\frac{5}{6}$, there should be some c for which $f(c) = C = 1$. To see what it is, we note that

$$\frac{c+3}{c^2+2} = 1$$

when $c^2 + 2 = c + 3$, that is, when

$$c^2 - c - 1 = 0.$$

The possibilities are

$$c = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1-\sqrt{5}}{2}.$$

The first lies between a and b, that is, between 0 and 2, as desired.

Example 8-2b. Let $f(x) = \frac{x+3}{x-1}$ with $a = 0$ and $b = 2$ and $C = 1$ as before. We have $f(a) = f(0) = -3$ and $f(b) = f(2) = 5$; $C = 1$ is between -3 and 5. Does there exist a number c, $(a < c < b)$ such that

$$f(c) = C = 1?$$

That is, is it possible that

$$\frac{c+3}{c-1} = 1?$$

This equation is equivalent to

$$c + 3 = c - 1,$$

which is equivalent to

$$3 = -1,$$

which is false. Hence, there is no possible solution c. This should not surprise us. The theorem assumes that f is continuous on $[a,b] = [0,2]$ here. However, for our f there is a discontinuity at 1.

Example 8-2c. Let $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ -1, & x = 0 \end{cases}$ with $a = 0$, $b = \frac{\pi}{2}$, $C = -\frac{1}{2}$

Although C is between $A = -1$ and $B = 0$ it is impossible to solve $f(c) = -\frac{1}{2}$. Of course, f is discontinuous at $a = 0$ so that the theorem does not apply. If we define f by

$$f : x \to \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

and choose $C = \frac{1}{2}$ we have better luck. In fact, C is between $A = 1$ and $B = f(\frac{\pi}{2}) = 0$; and

$$\frac{\sin c}{c} = \frac{1}{2}$$

does have a solution since f is continuous on $[0, \frac{\pi}{2}]$.

A second theorem about functions continuous on an interval $[a,b]$ guarantees the existence of maximum and minimum values.

THEOREM 8-2b. If f is continuous on $[a,b]$ there is at least one number c on $[a,b]$, $(a \leq c \leq b)$ where

(1) $f(x)$ is a maximum, M

and at least one number d, $(a \leq d \leq b)$ where

(2) $f(x)$ is a minimum, m.

Here (1) means that for all x on $[a,b]$, $f(x) \leq f(c)$ and (2) means that for all x on $[a,b]$, $f(d) \leq f(x)$.

A maximum or minimum value may occur between a and b or at an end-point. The following figures illustrate some of the possibilities.

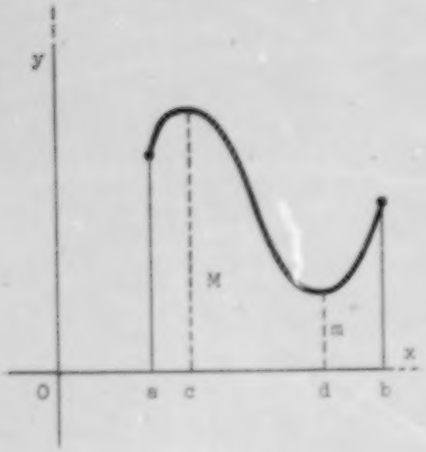


Figure 8-2c

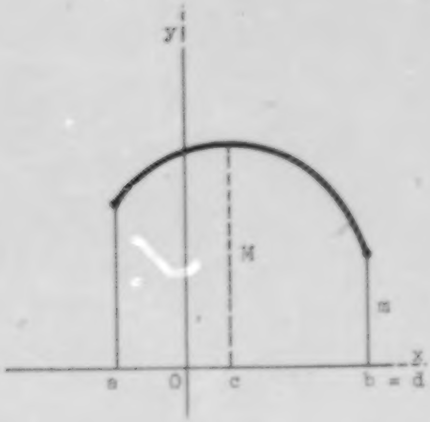


Figure 8-2d

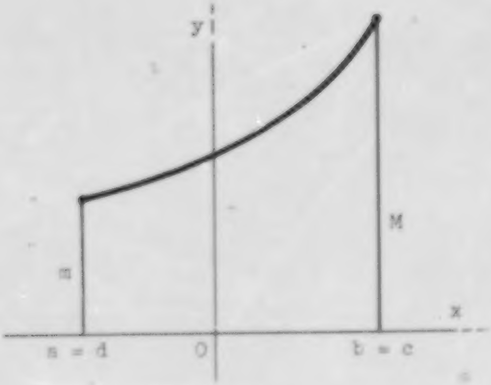


Figure 8-2e

We have the following theorem.

THEOREM 8-2c. If a maximum or minimum occurs between a and b (that is if it is not at an endpoint) and if $f'(x)$ exists there, then $f'(x) = 0$.

For definiteness let us consider the maximum $f(c)$ where $a < c < b$. If $f'(c) > 0$, there would be higher points nearby on the right. If $f'(c) < 0$, there would be higher points nearby on the left. Since both of these possibilities must be excluded, the only remaining possibility is $f'(c) = 0$.

The argument for the minimum value $f(d)$ is similar.

We give three examples of the use of this theorem.

Example 8-2d. Find M and m for $f(x) = \frac{1}{3}x^3 - x + 2$ on the interval $[-2,2]$. Since $f'(x) = x^2 - 1 = 0$ at $x = 1$ and $x = -1$, we should find $f(1) = \frac{4}{3}$ and $f(-1) = \frac{8}{3}$. At the endpoints, we have $f(-2) = \frac{4}{3}$ and $f(2) = \frac{8}{3}$. The minimum value $m = \frac{4}{3}$ and the maximum value $M = \frac{8}{3}$. Each occurs at an interior point and also at an endpoint.

If the interval were $[-3,3]$, we would have $f(-3) = -4$ and $f(3) = 8$. In this case, $m = -4$ and $M = 8$ and both the maximum value and the minimum value occur at the endpoints.

Example 8-2e. Let $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, and let $[a,b] = [0,\pi]$. As we know, f is not continuous on the whole interval $[a,b]$. Hence, the theorem does not apply. In fact, $f'(x) \neq 0$ at all interior points. There is a minimum value $m = 0$ at π. There is no maximum value.

If we change f so that $f(0) = 1$, f becomes continuous on $[0,\pi]$. $f(0) = 1$ is now M.

Example 8-2f. Let $f : x \to |x|$ and let $[a,b] = [-1,2]$. There is no point where $f'(x) = 0$. Turning to the endpoints $f(-1) = 1$ and $f(2) = 2$, we might be tempted to say that $m = 1$ and $M = 2$. Actually $f(0) = 0$ is the minimum value. It occurs at a point where $f'(x)$ does not exist.

Exercises 8-2

Apply the Intermediate Value Theorem 8-2a where possible. If the theorem does not apply, explain why not.

1. $f : x \to x^3 - 3x$ $\quad a = -1,\ b = 1,\ C = 0$
2. $f : x \to |x|$ $\quad a = -1,\ b = 2,\ C = \frac{3}{2}$
3. $f : x \to x^3 - 3x$ $\quad a = -1,\ b = 1,\ C = 1$
4. $f : x \to \frac{1}{x}$ $\quad a = -1,\ b = 1,\ C = 0$
5. $f : x \to \sin x$ $\quad a = 0,\ b = \frac{\pi}{2},\ C = \frac{1}{2}$
6. $f : x \to \sin x$ $\quad a = 0,\ b = \frac{\pi}{2},\ C = 2$
7. $f : x \to \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $\quad a = -1,\ b = 1,\ C = \frac{1}{2}$

Find m and M for each of the following functions on the interval indicated.

8. $f : x \to x^3 - 3x$ $\quad [-1,1]$
9. $f : x \to |x - 1|$ $\quad [0,2]$
10. $f : x \to x^3 - 3x$ $\quad [-2,2]$
11. $f : x \to \frac{1}{x}$ $\quad [-1,1]$
12. $f : x \to \sin x$ $\quad [0, \frac{\pi}{2}]$
13. $f : x \to \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $\quad [-1,1]$

8-3. The Mean Value Theorem

Consider the graph of $f : x \to x^2$ with the points $P(1,1)$ and $Q(2,4)$. The slope of the chord $\overline{PQ}$ is $\frac{4-1}{2-1} = 3$. The tangent to the graph at (x,x^2) has the slope $2x$. As we follow the arc from P to Q, this slope changes from 2 to 4, passing through the value 3 when $x = \frac{3}{2}$. At $R(\frac{3}{2},\frac{9}{4})$, therefore, the slope of the tangent is exactly equal to the slope of the chord and the tangent is parallel to the chord.

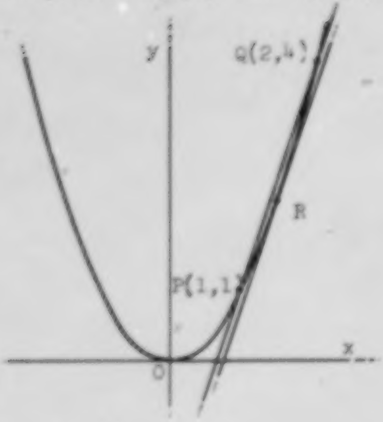


Figure 8-3a

We can generalize this idea. Consider the graph of any differentiable function f and let $P(a,f(a))$ and $Q(b,f(b))$ be two points on it. As we go from P to Q along the arc it seems reasonable to assume that somewhere between P and Q the tangent is parallel to the chord.

Let us consider other examples.

Example 8-3a. If $f : x \to x^3$ with $P(0,0)$ and $Q(2,8)$, the slope of $\overline{PQ} = \frac{8}{2} = 4$. At (x,x^3), $f'(x) = 3x^2$, which equals 4 when $x^2 = \frac{4}{3}$ and $x = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} \approx \frac{2}{3}(1.73) \approx 1.15$. The tangent at $R\left(\frac{2}{\sqrt{3}}, \frac{8}{3\sqrt{3}}\right)$ is parallel to $\overline{PQ}$. (Of course, $x = -\frac{2}{\sqrt{3}}$ is outside the interval $[0,2]$ from P to Q.)

Example 8-3b. With the same function $f : x \to x^3$ and the points $P(-1,-1)$ and $Q(1,1)$, the slope $(PQ) = 1$. Now

$$3x^2 = 1$$

and

$$x = -\frac{1}{\sqrt{3}} \text{ or } x = \frac{1}{\sqrt{3}}.$$

Thus we find two different points R, R' at which the tangent is parallel to the chord. (See Figure 8-3b.)

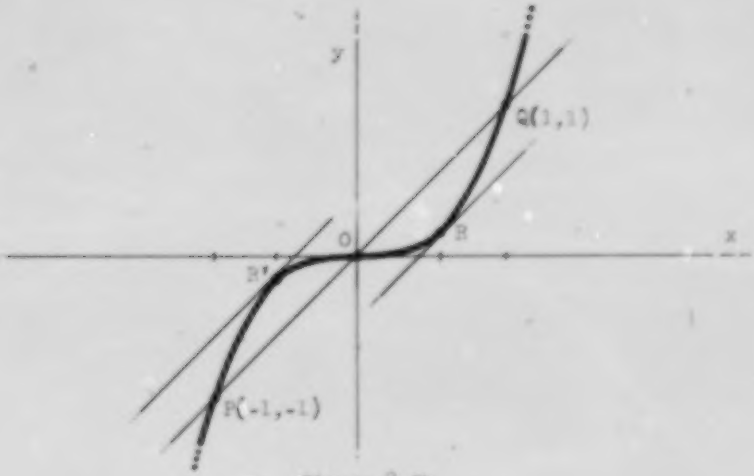


Figure 8-3b

Example 8-3c. Let $f : x \to |x|$ and let $P = (-1,1)$ and $Q = (2,2)$. Slope $(PQ) = \frac{1}{3}$. Is there any place R on the graph between P and Q for which $f'(x) = \frac{1}{3}$? The answer is "No." If $x > 0$, $f'(x) = 1$ and if $x < 0$, $f'(x) = -1$. At $x = 0$, there is no tangent.

This example shows that the principle that we are investigating may not hold if the function f fails to be differentiable at some point between P and Q.

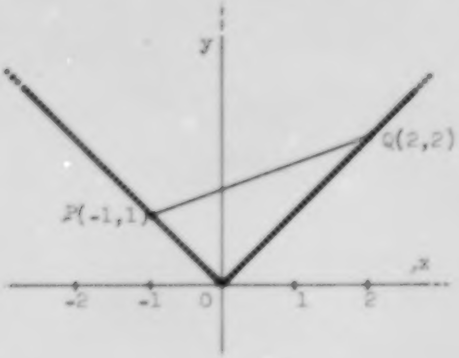


Figure 8-3c

Example 8-3d. Let $f : x \to \sqrt{x}$ with $P(0,0)$, $Q(4,2)$. The slope of $\overline{PQ} = \frac{1}{2}$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, $x \neq 0$, we have $\frac{1}{2\sqrt{x}} = \frac{1}{2}$ when $\sqrt{x} = 1$, that is, when $x = 1$. We note that f is not differentiable at $P(0,0)$ which is at one end of the chord. However, f is continuous at P.

__Example 8-3e.__ The graph of $f : x \to \sqrt{1 - x^2}$ is a semicircle with center at $(0,0)$ and radius 1. Any tangent is perpendicular to the radius. Hence, the slope at any point is the negative reciprocal of

$$\frac{\sqrt{1 - x^2}}{x} .$$

The derivative is therefore

$$f' : x \to - \frac{x}{\sqrt{1 - x^2}}$$

Notice that $f'(-1)$ and $f'(1)$ fail to exist. The tangent is vertical in each case. The function f is continuous at P and Q. If we choose $P(-1,0)$ and $Q(1,0)$, slope $(\overline{PQ}) = 0$. Is there a point R between P and Q at which $f'(x) = 0$? Of course, since $(0,1)$ is such a point.

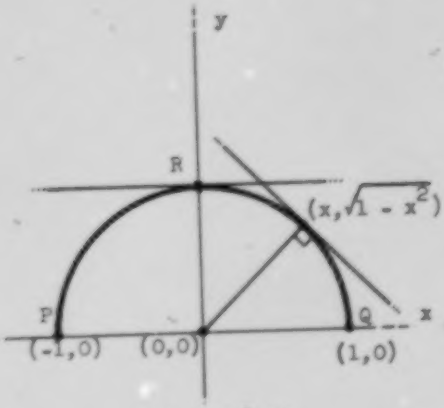


Figure 8-3d

We are now ready to state the theorem suggested by these examples.

__THEOREM 8-3.__ If f is differentiable for each x between a and b, $(a < x < b)$ and if f is continuous at $x = a$ and $x = b$, then there is at least one number c between a and b, $(a < c < b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

This is usually called the Mean Value Theorem because $\frac{f(b) - f(a)}{b - a}$ is the average or mean value of $f'(x)$ on the interval from a to b.

We shall not prove the Mean Value Theorem, but shall use it in the next section to draw certain important conclusions.

Exercises 8-3

1. Given $f : x \to x^2 + x$ and the points $P(0,0)$ and $Q(1,2)$. Find the point where the tangent is parallel to the chord $\overline{PQ}$.

2. Where must we choose Q on the graph of $x \to x^2$ so that with $P = (0,0)$ the chord $\overline{PQ}$ is parallel to the tangent at $(2,4)$?

3. Where is the tangent to $y = x^4$ parallel to the chord from $(-1,1)$ to $(2,16)$?

4. Suppose that you drive from Sacramento (elevation 200 feet) to Loggers Station Camp Ground (elevation 5480 feet). The map distance between the two points is exactly 100 miles. Was there some time during the trip when you were on a portion of road that had a slope of exactly 1%? Give your reason.

5. Suppose you drive from New York to Chicago, sometimes stopping and other times driving as fast as 70 miles per hour. Is there some time during the trip when your speed is 50 miles per hour? Give reasons.

6. Two cities are 200 miles apart. Starting from one you drive continually to the other in 4 hours, then stop.

   (a) Is there some place on the trip where your speedometer reads 50? Give reasons.

   (b) Is there some place on the trip where your acceleration was 0? Give reasons.

7. Given $f : x \to \frac{1}{x}$ is there a point where the tangent is parallel to the chord $\overline{PQ}$ where $P(1,1)$, $Q(2, \frac{1}{2})$? If so, find it.

8. Given $f : x \to \frac{1}{x}$ is there a point where $f'(x)$ is equal to the slope of $\overline{PQ}$ where $P(-1,-1)$, $Q(1,1)$? Explain.

9. Let [illegible] $: x \to \frac{1}{x}$ if $x > 0$ and let $f(0) = 0$. With $P(0,0)$ and $Q(1,1)$ is there a point between P and Q at which $f'(x) = \text{slope}(\overline{PQ})$? Explain.

8-4. Applications of the Mean Value Theorem

If we use the Mean Value Theorem (8-3) we can now prove certain results that we have previously taken for granted.

THEOREM 8-4a. If $f'(x) > 0$ when x is between a and b $(a < x < b)$ and if $f(x)$ is continuous at a and b, then $f(x)$ increases uniformly on the interval $a \leq x \leq b$.

(We include the requirement that f be continuous at a and b in order to be able to apply the Mean Value Theorem.)

By this we mean that for all numbers x_1 and x_2 such that $a \leq x_1 < x_2 \leq b$, $f(x_1) < f(x_2)$. (See Figure 8-4a, where x_1 may coincide with a or x_2 with b.)

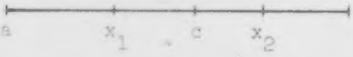


Figure 8-4a

Proof. According to the Mean Value Theorem

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad (x_1 < c < x_2).$$

Since this means that

$$a < c < b$$

it follows that $f'(c) > 0$. Hence,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$$

and
$$f(x_2) > f(x_1);$$

that is,
$$f(x_1) < f(x_2).$$

In the same way we can easily prove the following theorem.

THEOREM 8-4b. If $f'(x) < 0$ when $a < x < b$ and if f is continuous at a and b, then f decreases uniformly on the interval $a \leq x \leq b$.

What can we conclude if $f'(x) = 0$ for all x between a and b? In this case, for all x_1 and x_2 such that $a \leq x_1 < x_2 \leq b$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$$

$$f(x_2) - f(x_1) = 0,$$

and
$$f(x_2) = f(x_1).$$

Since this is true for all x_1 and larger x_2 on the interval $[a,b]$, f is a constant function on this interval.

THEOREM 8-4c. If $f'(x) = 0$ for $a < x < b$ and if f is continuous at a and b, then f is a constant function on $[a,b]$.

What can we conclude if we know that the derivative f' increases (decreases) uniformly on an interval $[a,b]$?

Let c be any number between a and b $(a < c < b)$. If $x > c$

$$\frac{f(x) - f(c)}{x - c} = f'(d)$$

where $c < d < x$ (see Figure 8-4b).
But $f'(d) > f'(c)$. Hence,

$$\frac{f(x) - f(c)}{x - c} > f'(c).$$

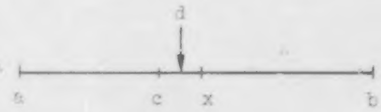


Figure 8-4b

Then

$$f(x) - f(c) > f'(c)\,(x - c)$$

and

$$f(x) > f(c) + f'(c)\,(x - c).$$

This means that to the right of c the graph of f lies above the tangent at c (Figure 8-4c).

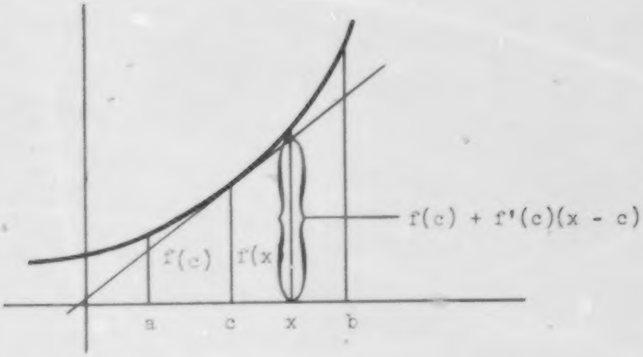


Figure 8-4c

Similarly, if $x < c$

$$\frac{f(c) - f(x)}{c - x} = f'(e), \quad x < e < c.$$

Hence,

$$\frac{f(c) - f(x)}{c - x} < f'(c)$$

$$f(c) - f(x) < f'(c)\,(c - x)$$

and $$f(x) > f(c) - f'(c)\,(c - x)$$

or finally $$f(x) > f(c) + f'(c)\,(x - c).$$

Again the graph of f lies above the tangent at c.

If a function f has the property that its graph lies above each tangent whose point of contact lies within an interval $[a,b]$ (except at the point of contact), then we say that f is convex on $[a,b]$.

If we replace "above" by "below" in this statement, "convex" is replaced by "concave."

THEOREM 8-4d. If f' increases (decreases) uniformly on an interval $[a,b]$, f is convex (concave) on $[a,b]$.

Exercises 8-4

1. What does the Mean Value Theorem become if $f(a)$ and $f(b)$ are both equal to zero? The result is called Rolle's Theorem.

2. Suppose that for a function f we know that $f'' > 0$ on an interval $a < x < b$. Show from the theorems of this section that f is convex on the interval.

3. If $f'(x) > 0$ for $a < x < b$ and $f'(x) < 0$ for $b < x < c$ while $f'(b) = 0$, use the theorems of this section to draw an appropriate conclusion.

4. Suppose that $f'(x) = g'(x)$ for all x on an interval $[a,b]$. Show that $f(x) - g(x) = C$ on this interval where C is a constant.

   Hint: Assume that $[f(x) - g(x)]' = f'(x) - g'(x)$ and use Theorem 8-4c.

   This is the important Constant Difference Theorem of Section 7-3.

8-5. Sums and Multiples

The remaining sections of this chapter discuss methods for differentiating various combinations of known functions. In this section we examine sums and multiples of functions.

Functions which are the sum of other functions have been previously encountered many times. For example, the graph of

$$f : t \to 3 \cos \pi t + 4 \sin \pi t$$

was obtained in Chapter 3 by adding the corresponding ordinates (Figure 8-5a) of the two functions

$$u : t \to 3 \cos \pi t \quad \text{and} \quad v : t \to 4 \sin \pi t$$

at each value of t.

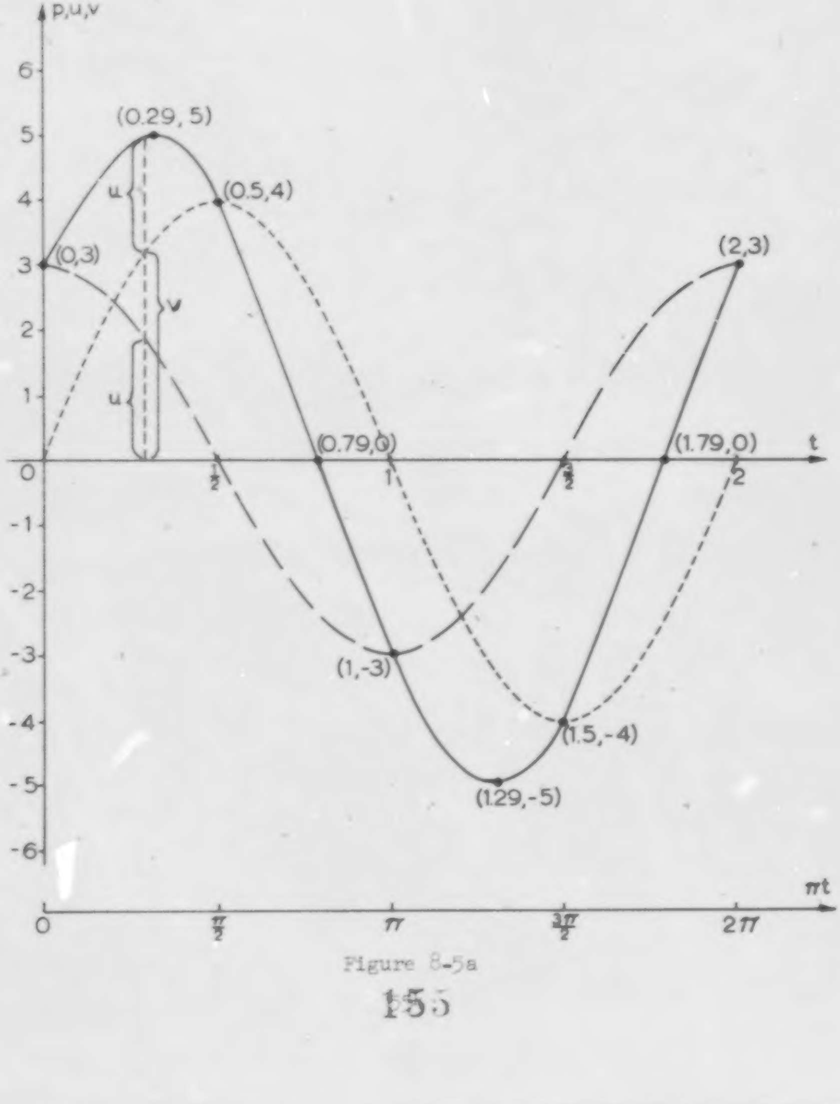


Figure 8-5a

Here we say that f is the sum of the two functions u and v and write

$$f = u + v.$$

This means that for each t, the values $f(t)$, $u(t)$ and $v(t)$ are related by

$$f(t) = u(t) + v(t).$$

The difference of two functions is defined analogously; for example,

$$f = u - v$$

if, for each x, the values $f(x)$, $u(x)$ and $v(x)$ are related by

$$f(x) = u(x) - v(x).$$

To be more concrete, if

$$f : x \rightarrow 2 \sin 3x - 3 \cos 3x$$

we can write $f = u - v$, where

$$u : x \rightarrow 2 \sin 3x$$

$$v : x \rightarrow 3 \cos 3x.$$

The function $u : x \rightarrow 2 \sin 3x$ is a multiple of the function

$$g : x \rightarrow \sin 3x$$

in the sense that the values $u(x)$ and $g(x)$ are related by the equation $u(x) = 2g(x)$. The graph of u is obtained from the graph of g by multiplying the corresponding ordinate of the graph of g by 2. (See Figure 8-5b.)

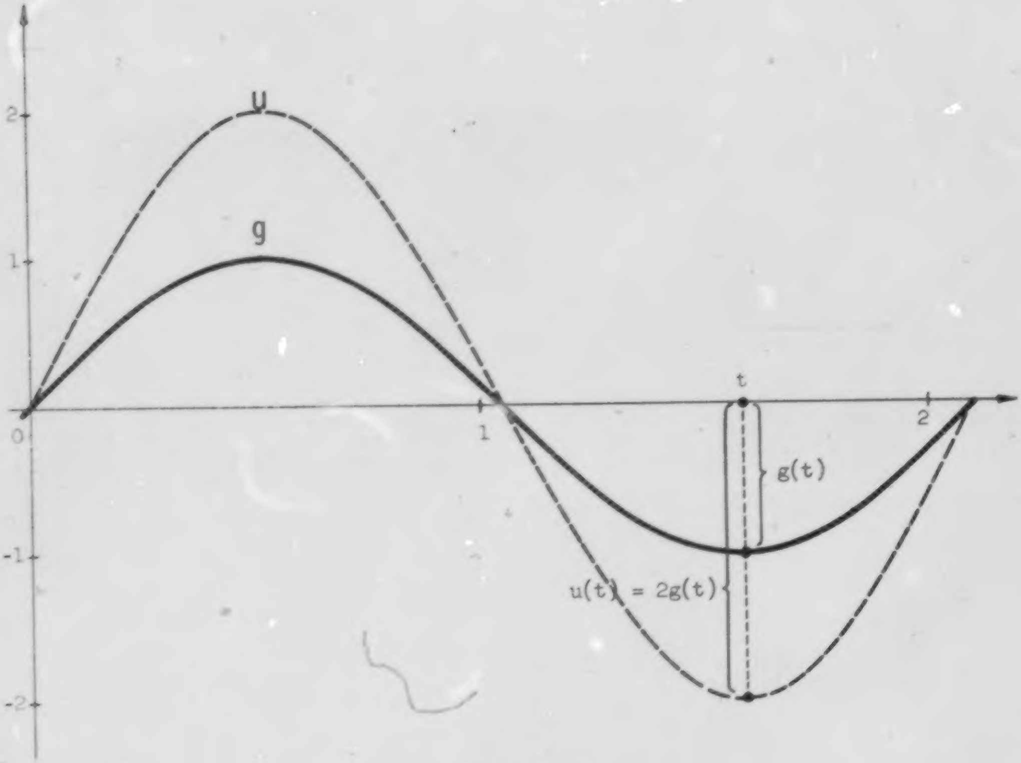


Figure 8-5b

$u = 2g$, where $u : x \to 2 \sin 3x$, $g : x \to \sin 3x$.

The basic rules for derivatives of sums and multiples are easily obtained and simply stated:

(1) $$\boxed{\text{if } f = u + v, \text{ then } f' = u' + v';}$$

and, for any constant a,

(2) $$\boxed{\text{if } f = ag, \text{ then } f' = ag'.}$$

For example, if $f = u + v$, where

$$u : t \to 3 \cos \pi t \text{ and } v : t \to 4 \sin \pi t,$$

then

$$f' = u' + v';$$

that is, for each t,

$$\begin{aligned} f'(t) &= u'(t) + v'(t) \\ &= -3\pi \sin \pi t + 4\pi \cos \pi t. \end{aligned}$$

We also made use of (2). For example, that $u'(t) = -3\pi \sin \pi t$ makes use of the fact that $D(3 \sin \pi t) = 3D(\sin \pi t)$.

We can use the concept of approximation along the tangent line to the graph of a function to show that (1) and (2) hold. For example, suppose $f = u + v$, where u and v are each differentiable at a. For the best linear approximation to the graphs of u and v respectively we have

$$\begin{aligned} u(x) &\approx u(a) + u'(a)(x - a), \\ v(x) &\approx v(a) + v'(a)(x - a), \end{aligned} \tag{3}$$

if x is close to a. Adding, we have

$$u(x) + v(x) \approx u(a) + v(a) + (u'(a) + v'(a))(x - a). \tag{4}$$

Now we use the assumption that $f = u + v$ to obtain

$$f(x) \approx f(a) + (u'(a) + v'(a))(x - a).$$

For $x \neq a$ we subtract $f(a)$ from both sides and divide by $x - a$ to get

$$\frac{f(x) - f(a)}{x - a} \approx u'(a) + v'(a).$$

We take the limit as x approaches a to obtain

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = u'(a) + v'(a).$$

We conclude that

$$f'(a) = u'(a) + v'(a).$$

The easier intuitive argument which establishes that if $f = ag$ then $f' = ag'$ was given in Chapter 6.

We can combine results (1) and (2) to differentiate $f = u - v$, for we can write

$$f = u + w, \quad \text{where} \quad w = (-1)v,$$

so that

$$f' = u' + w' \quad \text{and} \quad w' = (-1)v' = -v'.$$

Thus, as we should expect:

$$f' = u' - v' \quad \text{if} \quad f = u - v. \tag{5}$$

Example 8-5a. Find the derivative of $f : x \to x - \sin x$ and discuss its graph in the interval $-2\pi \leq x \leq 2\pi$.

We can let $u : x \to x$ and $v : x \to \sin x$, so that $f = u - v$. Since, from (5), $f' = u' - v'$ and

(6) $$u' : x \to 1, \quad v' : x \to \cos x,$$

we have the result

$$f'(x) = 1 - \cos x.$$

For all x, $f'(x) \geq 0$, since $\cos x \leq 1$. This tells us that f is an increasing function for all x. Furthermore, the graph of f has a horizontal tangent at each of the points $(-2\pi, f(-2\pi))$, $(0, f(0))$ and $(2\pi, f(2\pi))$ since

$$f'(-2\pi) = f'(0) = f'(2\pi) = 0.$$

Let us differentiate again. Since

$$f' = u' - v'$$

we can apply (5) with f, u and v replaced by f', u' and v' to obtain the result

$$f'' = u'' - v''.$$

Making use of (6), we have

$$u'' : x \to 0 \text{ and } v'' : x \to -\sin x$$

so that

$$f'' : x \to \sin x.$$

The function f'' is nonnegative in the intervals

(7) $$-2\pi \leq x \leq -\pi, \text{ and } 0 \leq x \leq \pi$$

and nonpositive in the intervals

(8) $$-\pi \leq x \leq 0 \text{ and } \pi \leq x \leq 2\pi.$$

Thus, the graph of f is convex in the intervals of (7) and concave in the intervals of (8).

The graph of f (Figure 8-5c) is obtained by making use of this information and plotting a few points.

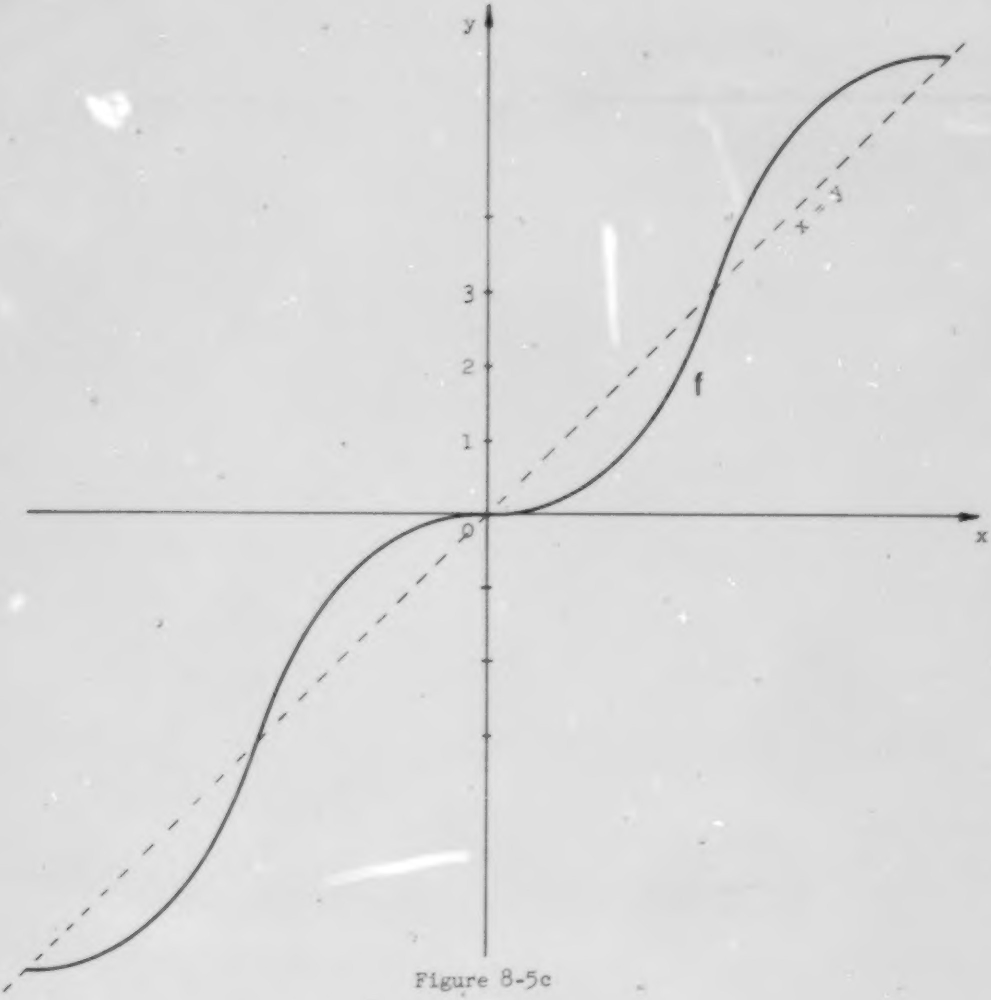


Figure 8-5c

$y = x - \sin x$

Example 8-5b. Suppose a particle moves along a horizontal line so that its distance from the origin at time $t > 0$ is given by $s = t + \frac{1}{t}$. Discuss the motion.

If t is close to 0, then s is nearly equal to and slightly larger than $\frac{1}{t}$, which is very large. If t is very large, then $\frac{1}{t}$ is very small, so that s is nearly equal to but slightly larger than t. Geometrically these observations mean that for $t > 0$ the graph of $s = t + \frac{1}{t}$ approaches the s-axis as t approaches 0 and approaches the line given by $s = t$ as t becomes large. In other words, the vertical line given by $t = 0$ is an asymptote for the graph of $s = t + \frac{1}{t}$ as t approaches 0, while the line given by $s = t$ is an asymptote for the graph as t grows large without bound through positive values.

The derivative of $t \to t + \frac{1}{t}$ can be obtained using the sum formula (1). We have

$$D(t + \frac{1}{t}) = Dt + D(\frac{1}{t}) = Dt + D(t^{-1}).$$

Since $Dt = 1$ and $Dt^{-1} = -1t^{-2} = -\frac{1}{t^2}$, we conclude that

$$D(t + \frac{1}{t}) = 1 - \frac{1}{t^2} .$$

The value of the derivative $t \to s' = 1 - \frac{1}{t^2}$ is the velocity at time t. Since $s' < 0$ if $t < 1$ and $s' > 0$ if $t > 1$, the function $t \to t + \frac{1}{t}$ decreases in the interval $0 < t < 1$ and increases in the interval $t > 1$. When $t = 1$, the value of the derivative is 0 and 2 is the minimum value of s. This means that the particle moves toward the origin as t increases from 0 to 1, is closest to the origin when $t = 1$ and then moves steadily away from the origin.

The second derivative is obtained by using the difference formula (5) and the power formula:

$$D\left(1 - \frac{1}{t^2}\right) = D1 - D(t^{-2}) = \frac{2}{t^3} .$$

Thus, the acceleration is always positive (since t is positive), is very large when t is close to 0, and approaches 0 as t grows large without bound. The second derivative

$$t \to \frac{2}{t^3}$$

tells us that the graph of $s = t + \frac{1}{t}$ is convex. The graph is given in Figure 8-5d.

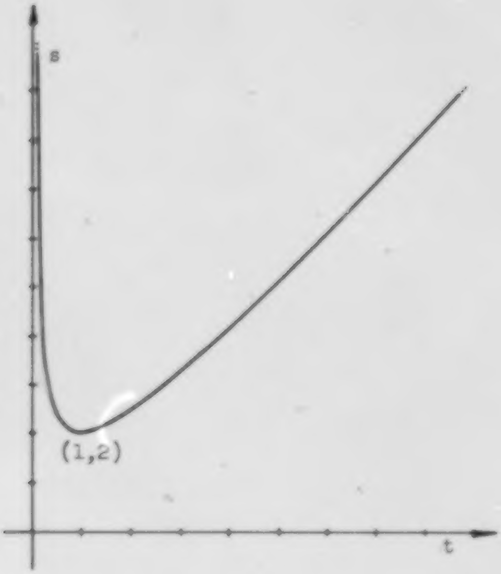


Figure 8-5d

$s = t + \frac{1}{t}$

Exercises 8-5

1. Find the derivatives of each of the following

(a) $y = x^{1/3} - 3x^{-2/5}$

(b) $y = x^2 + 2 \sin x$

(c) $y = (3x^2 + 1)(x^4 + 1)$

(d) $y = (1 - 2x)(\frac{1}{x^2} + \frac{1}{x})$

(e) $y = e^x + e^{2x} + \cos x$

(f) $y = \sqrt{x} - 3e^{-x}$

(g) $y = x + \log_e x^2 - 2 \log_e x$

(h) $y = x^e + e^x$

2. Sketch graphs of $f : x \to \sqrt{x} + \frac{1}{x}$, $u : x \to \sqrt{x}$ and $v : x \to \frac{1}{x}$ for $0 < x \leq 1$. What is the equation of the tangent line to each at the point where $x = \frac{1}{2}$? How are these tangent lines related?

3. (a) At what points on the graph of

$$y = \sin x - \sqrt{3} \cos x$$

is the tangent line horizontal?

(b) At what points on the graph of

$$y = 2^x - 2x$$

is the tangent line perpendicular to the line whose equation is $y = 3x + 2$?

(c) Suppose the tangent lines to the graphs of $y = 5f(x)$ and $y = 7f(x)$ are parallel and nonvertical at the point where $x = a$. Show that these tangent lines must be horizontal.

(d) Show that if u and v are differentiable at $x = a$ and the graphs of $f : x \to u(x) + 3v(x)$ and $g : x \to u(x) - 11v(x)$ have the same slope at the point where $x = a$ then v has a horizontal tangent at $(a, v(a))$.

4. Show that if a and b are constants then

$$D(au + bv) = a\,Du + b\,Dv.$$

5. Analyze

   (i) increase-decrease,

   (ii) convexity-concavity and

   (iii) asymptotes (if any)

   for each of the following functions on the interval given. Sketch graphs.

   (a) $f : x \to x - \cos x, \quad 0 \leq x \leq 2\pi$

   (b) $f : x \to e^x - 2x, \quad 0 \leq x \leq 1$

   (c) $f : t \to t^2 + \frac{3}{t}, \quad 0 < t$

   (d) $f : x \to x^2 - \sqrt{2x}, \quad 0 \leq x \leq 2$

6. (a) Show that $D\displaystyle\int_x^b f(x)dx = -f(x)$.

   (b) Find $D\displaystyle\int_x^b e^{-t^2}\, dt$.

7. Show that the acceleration of a particle whose equation of motion is $s(t) = 2 \cos t + t^2$ is always nonnegative.

8. Suppose you know only that the rules of this section hold and that $Dx^n = nx^{n-1}$. Can you find the derivative of a polynomial?

9. Consider $g : x \to |x + 2| - |3 - x|$.

   (a) Sketch the graph of g.

   (b) Define $g(x)$ explicitly in terms of linear functions for all real x.

   (c) For what values of x is the derivative not defined?

10. (a) $1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + x^2, \quad 0 \leq x \leq 2$

    (Hint: Put $f($[illegible]$) = e^x - \left(1 + x + \frac{x^2}{2}\right)$ and find the minimum of f. Proceed in a similar manner for the right-hand side).

    (b) Show that if $u(a) \leq v(a)$ and $u'(x) \leq v'(x)$ for $x \geq a$ then $u(x) \leq v(x)$ for $x \geq a$. (Hint: Consider $f = v - u$.)

    (c) Show that if $u(a) \leq v(a)$, $u'(a) \leq v'(a)$ and $u''(x) \leq v''(x)$ for $x \geq a$ then $u(x) \leq v(x)$ for $x \geq a$. (Hint: Use (b) twice. First show that $u'(x) \leq v'(x)$ when $a \leq x$.)

11. (a) Show that if $y = u$ and $y = v$ are solutions to the equation $y'' - 3y' + 6y = 0$, then so is $y = 3u + 8v$.

    (b) Show that $y = e^x + e^{-x}$ and $y = e^x - e^{-x}$ are each solutions to the equation $y'' = y$. If α and β are constants is

    $$y = \alpha(e^x + e^{-x}) + \beta(e^x - e^{-x})$$

    also a solution to $y'' = y$?

12. Suppose $u(x) = v(x) + ax + b$, where a and b are constants.

    (a) What is $u'(x) - v'(x)$?

    (b) Show that $u'' = v''$.

    (c) Prove the following converse: If $u'' = v''$ then $u - v$ is a linear function. (Hint: Use the Constant Difference Theorem twice.)

13. Suppose u and v are continuous at $x = a$. Is $f = 2u - 3v$ also continuous at $x = a$?

14. Suppose $f = u + v$ and f is differentiable and thus continuous at $x = a$. Must u and v also be differentiable and thus continuous at $x = a$? If so, why? If not, give an example.

8-6. Products

Each value of the function

$$f : x \to xe^x$$

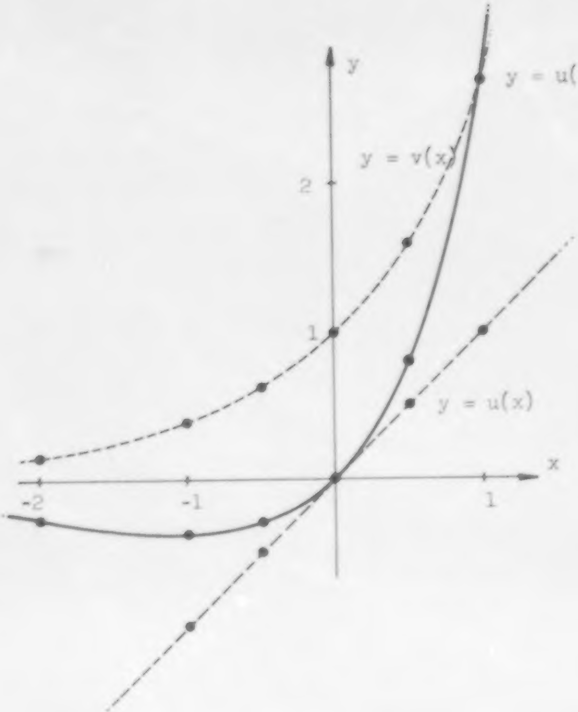
is just the product of the corresponding values of the two functions

$$u : x \to x \quad \text{and} \quad v : x \to e^x;$$

that is, for each x,

$$f(x) = u(x)v(x).$$

This relationship can be used to obtain the graph of f from the graphs of u and v, for the ordinate of a point on the graph of f is the product of the corresponding ordinates of the graphs of u and v. (See Figure 8-6a.)



x	u(x)	v(x)	f(x) = u(x)v(x)
-2	-2	0.14	-0.28
-1	-1	0.38	-0.38
-0.5	-0.5	0.61	-0.30
0	0	1	0
0.5	0.5	1.6	0.8
1	1	2.7	2.7

Figure 8-6a

$y = xe^x$

In general, we say that the function f is the product of the two functions u and v and write

$$f = uv$$

if for each x the values $f(x)$, $u(x)$ and $v(x)$ are related by

(1) $$f(x) = u(x)v(x).$$

A formula for the derivative of $f = uv$ in terms of the derivatives of u and v can be obtained by using tangent line approximations. Suppose u and v are each differentiable at $x = a$ so that, if we take x close to a, we have the best linear approximations

$$u(x) \approx u(a) + u'(a)(x - a)$$

$$v(x) \approx v(a) + v'(a)(x - a).$$

For the product we get

$$u(x)v(x) \approx u(a)v(a) + [u(a)v'(a) + v(a)u'(a)](x - a) + u'(a)v'(a)(x - a)^2.$$

Since $f = uv$ we can rewrite this as

$$f(x) \approx f(a) + [u(a)v'(a) + v(a)u'(a)](x - a) + u'(a)v'(a)(x - a)^2$$

so that, for $x \neq a$

$$\frac{f(x) - f(a)}{x - a} \approx [u(a)v'(a) + v(a)u'(a)] + u'(a)v'(a)(x - a).$$

It follows that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = u(a)v'(a) + v(a)u'(a).$$

Thus, we obtain the product rule:

(2) $$f'(a) = u(a)v'(a) + v(a)u'(a).$$

This formula is sometimes written in the form

(3) $$(uv)' = uv' + vu'$$

or

(4) $$D(uv) = uDv + vDu,$$

or expressed in words:

(5) The derivative of the product of two functions is the first times the derivative of the second plus the second times the derivative of the first.

For example, $f : x \to x \log_e x$ is the product of

$$u : x \to x \text{ and } v : x \to \log_e x.$$

Since $u'(x) = 1$ and $v'(x) = \frac{1}{x}$, the product rule gives

$$f'(x) = x \cdot \frac{1}{x} + (\log_e x) \cdot 1 = 1 + \log_e x.$$

As another example we consider the function

$$f : x \to e^{3x} \sin 2x,$$

which is the product of

$$u : x \to e^{3x} \text{ and } v : x \to \sin 2x.$$

The product rule gives

$$f'(x) = e^{3x} \cdot (2 \cos 2x) + (\sin 2x)(3e^{3x}).$$

Example 8-6a. Locate the intervals of increase and decrease, convexity and concavity for the graph of the function

$$f : x \to xe^x.$$

The function f is the product of

$$u : x \to x \text{ and } v : x \to e^x,$$

so that

$$\begin{aligned} f'(x) &= u(x)v'(x) + v(x)u'(x) \\ &= x\, e^x + e^x \cdot 1 \\ &= (x + 1)e^x. \end{aligned}$$

This will be positive for $x > -1$ and negative for $x < -1$ so that the graph of f falls until it reaches $(-1, -\frac{1}{e})$ and rises after that point.

The function $f' : x \to (x + 1)e^x$ is the product of

$$u : x \to x + 1 \text{ and } v : x \to e^x$$

so the product rule gives

$$\begin{aligned} f''(x) &= u(x)v'(x) + v(x)u'(x) \\ &= (x + 1)e^{x} + e^{x} \cdot 1 \\ &= (x + 2)e^{x}. \end{aligned}$$

We conclude from this that the graph of f is concave for $x < -2$ and convex for $x > 2$. An extension of our sketch (Figure 8-6a) should reflect these conclusions. We should also note that as x moves far to the left $f(x) = xe^{x}$ approaches 0; that is, the negative x-axis is an asymptote for the graph of f as x grows large without bound through negative values.

__Example 8-6b__. Show that if $f : x \rightarrow e^{ax} \sin bx$, then $f''(x) - 2af'(x) + (a^2 + b^2)f(x) = 0$.

The product rule gives

$$\begin{aligned} f'(x) &= e^{ax}\, D(\sin bx) + (\sin bx)\, D(e^{ax}) \\ &= e^{ax}(b \cos bx) + (\sin bx)(ae^{ax}) \\ &= e^{ax}[b \cos bx + a \sin bx]. \end{aligned}$$

Again we use the product rule (as well as the sum rule) to obtain

$$\begin{aligned} f''(x) &= e^{ax}\, D[b \cos bx + a \sin bx] + [b \cos bx + a \sin bx]D(e^{ax}) \\ &= e^{ax}[-b^2 \sin bx + ab \cos bx] + [b \cos bx + a \sin bx]ae^{ax} \\ &= e^{ax}[(a^2 - b^2)\sin bx + 2ab \cos bx]. \end{aligned}$$

Therefore,

$$\begin{aligned} f''(x) - 2af'(x) + (a^2 + b^2)f(x) &= e^{ax}[(a^2 - b^2)\sin bx + 2ab \cos bx] \\ &\quad - 2ae^{ax}[b \cos bx + a \sin bx] \\ &\quad + e^{ax}[(a^2 + b^2)\sin bx] \\ &= e^{ax}[(a^2 - b^2 - 2a^2 + a^2 + b^2)\sin bx] \\ &\quad + e^{ax}[(2ab - 2ab)\cos bx] \\ &= 0. \end{aligned}$$

Example 8-6c. Suppose f is a polynomial function and that a is a zero of f. Show that the multiplicity of a is greater than 1 if and only if a is a zero of f'.

If the multiplicity of a exceeds 1 then $(x - a)^2$ is a factor of $f(x)$; that is

$$f(x) = (x - a)^2 q(x),$$

where q is a polynomial function. Applying the product rule we have

$$f'(x) = (x - a)^2 q'(x) + q(x) \cdot 2(x - a),$$

so that indeed

$$f'(a) = 0.$$

If the multiplicity of a is 1, then

$$f(x) = (x - a)g(x), \quad \text{where} \quad g(a) \neq 0.$$

The product rule gives

$$f'(x) = (x - a)g'(x) + 1 \cdot g(x),$$

so that

$$f'(a) = g(a) \neq 0.$$

In other words, if the multiplicity of a is 1 then a cannot also be a zero of f'.

Exercises 8-6

1. Let $y_1 = a_1 + m_1(x - a)$ be the equation of the tangent line to the graph of $u : x \to x^2$ at (a,a^2) and $y_2 = a_2 + m_2(x - a)$, the equation of the tangent line to the graph of $v : x \to x^3$ at (a,a^3).

   (a) Find a_1, m_1, a_2, m_2.

   (b) Form the product of the expressions for y_1 and y_2, and omit the term involving $(x - a)^2$. The resulting expression is linear in $(x - a)$ and hence defines a line. Show that this line is the tangent line to $uv = f : x \to x^5$ at the point (a,a^5).

2. Find the derivative of f, where $f(x)$ equals

   (a) $x(2x - 3)$

   (b) $(4x - 2)(4 - 2x)$

   (c) $(x^2 + x + 1)(x^2 - x + 1)$

   (d) $\sqrt{x}\,(ax + b)^3$

   (e) $\frac{1}{x} \cdot \sqrt{x}$

   (f) $\frac{1}{x} \cdot (5x + 2)$

   (g) $x\,e^x$

   (h) $x^{7/2}$, $x > 0$

   (i) $3x^4 - \frac{1}{\sqrt{x}}$

   (j) $3x^2(x^2 - 5)$

   (k) $\sqrt{x}\cos 2x$

   (ℓ) $e^{3x}\sin(x + 1)$

   (m) $x^2\log_e x$

   (n) $(x - 1)^{1/2}\,e^{-x}$

   (o) $x\int_0^x e^{-t^2}\,dt$

   (p) $e^x\int_1^x \frac{\sin t}{t}\,dt$

   (q) $x\,e^x\sin x$

   (r) $(\log_e x)(4x^2 + 2x)(\cos 2x)$

   (s) $2\sin x\cos x$

   (t) $x\,e^x\log_e(2x + 1)(\sin x)$

   (u) $x^2\,2^x$

   (v) $x\log_2(3x + 1)$

   (w) $x^e\,e^x$

3. Evaluate

(a) $D(3x^2 + 5x - 1)^2$

(b) $D(3 - 5x)^3$

(c) $D(3 - 5x)^4$

(d) $D(x(\sqrt{x} - 1)^2)$

(e) $D(x + \frac{1}{x})^2$

(f) $D\left(\frac{x^{3/2}}{3} - \frac{x^{1/2}}{2} + x^{-1/2}\right)$

(g) $D\left(4\sqrt{x^3} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

(h) $D(e^x \sin(1 - 2x))$

(i) $D(\sqrt{x} \log_e x)$

(j) $D(x^\pi \pi^x)$

(k) $D(x^2 \cos x)$

(ℓ) $D(\sin x \log_e x)$

(m) $D\left(\frac{\log_e x}{x}\right)$

4. (a) Suppose $f(x) = [u(x)]^2$. Show that $f'(x) = 2u(x)u'(x)$. (Hint: Use the Product Rule.)

(b) Show that $D[u(x)]^3 = 3[u(x)]^2u'(x)$.

(c) Show that $D[u(x)]^4 = 4[u(x)]^3u'(x)$.

(d) Make a conjecture about $D[u(x)]^n$.

5. Use the results of Number 4 to find y' if

(a) $y = \sin^2 x$

(b) $y = \cos^3($[illegible]$x)$

(c) $y = (\log_e x)^2$

(d) $y = (e^x)^4$

(e) $y = (x^2 + 1)^2$

(f) $y = \sin^3 (2x - 1)$

(g) $y = \left(\int_1^x \sin t^2 \, dt\right)^4$

6. Combine the method of Number 4 with the **Product Rule** to find $\frac{dy}{dx}$ if

(a) $y = x^2(x^2 + 1)^2$

(b) $y = (x + 1)^3(x^2 - x + 1)$

(c) $y = (ax^2 + bx + c)(dx^2 + ex + f)$

(d) $y = (\cos^2 x)\sin 2x$

(e) $y = e^x \sin^2 (ax + b)$

(f) $y = (x\int_0^x e^{t^2}\, dt)^2$

(g) $y = x^3[\log_e (x + 1)]^3$

7. For each of the following functions, find the intervals of increase (or decrease) and convexity (or concavity). Sketch graphs over the intervals indicated.

(a) $y = x \log_e x, \quad 0 < x \le e$ (c) $y = \sin^3 x, \quad 0 \le x \le 2\pi$

(b) $y = \frac{1}{x} \log_e x, \quad 0 < x \le e^2$ (d) $y = x^2 \log_e x, \quad 0 < x \le 8$

8. Show that each of the following is an increasing function

(a) $x \to \sqrt{x}\, e^x, \quad x > 0$

(b) $x \to \frac{e^x}{x}, \quad x \ge 1$

(c) $x \to \frac{e^x}{x^\alpha}, \quad x \ge \alpha > 0$

(d) $x \to x \sin x, \quad 0 \le x \le \frac{\pi}{2}$

9. Show that if $f(x) = (x - a)^2 g(x)$ where g is differentiable and $g(a) \neq 0$, then $f'(a) = 0$.

10. Show that if a is a zero of the polynomial function f of multiplicity greater than 2 then $f''(a) = 0$. If $f''(a) = 0$ must it be true that a is a zero of f of multiplicity greater than 2?

11. (a) Show that if $y = e^{ax} \cos bx$ then $y'' - 2ay' + (a^2 + b^2)y = 0$.

(b) Show that if $y = x^2 e^x + 2xe^x$ then $y''' - 3y'' + 3y' - y = 0$.

12. (a) Show that

$$(uv)'' = uv'' + 2u'v' + u''v.$$

(b) Use (a) to find the second derivative of

$$f : x \to x^2 \cos x.$$

(c) What is $(uv)'''$?

(d) Does (c) lead you to a conjecture about the nth derivative of uv?

8-7. Composite Functions

The function $f : x \rightarrow \sqrt{x^2 + 1}$ is not a polynomial, circular, power, exponential or logarithm function; nor is it a sum or product of such functions. The verbal description of f can give a clue as to how to treat such a function. Verbally, the rule for f is

(1) "the square root of the quantity x squared plus one."

In other words, first calculate the quantity $x^2 + 1$, and then take the square root of the result. The operation defined by f is composed of two simpler operations, finding $x^2 + 1$ and taking square roots. In this and the next two sections we discuss functions which are compositions of other functions.

The statement (1) can be translated into a symbolic form which will display the fact that $f : x \rightarrow \sqrt{x^2 + 1}$ is composed of the two operations, $x \rightarrow x^2 + 1$ and taking square roots. Let $g(x) = u = x^2 + 1$ and $h(u) = \sqrt{u}$, so that

$$f(x) = h(g(x)).$$

To evaluate $f(x)$ we first evaluate $g(x)$, then evaluate $h(g(x))$. For example, if $x = 3$, then

$$u = g(3) = 3^2 + 1 = 10$$

and

$$f(3) = h(g(3)) = h(10) = \sqrt{10}.$$

In general, we say that a function f is a composition of the two functions h and g, if whenever $f(x)$ is defined, so are $g(x)$ and $h(g(x))$; and then

$$f(x) = h(g(x)).$$

The idea of composition has been previously used implicitly. For example, the function

$$f : x \rightarrow \sin (2x + 3)$$

is a composition of the functions $h : u \rightarrow \sin u$ and $g : x \rightarrow u = 2x + 3$; that is,

$$f(x) = h(g(x)).$$

Also, use has been made of the fact that the general exponential function

$f : x \to a^x$ is a composite function since we can write $a = e^{\alpha}$. If $h : u \to e^u$ and $g : x \to \alpha x = u$, then

$$f : x \to a^x = h(g(x)) = e^{\alpha x}.$$

Facility with composite functions depends upon ability to write complicated expressions as composites of simpler expressions. Some examples and practice exercises are provided to help you develop skill at doing this.

Example 8-7a. Express $x \to \sin \sqrt{x}$ as the composite of simpler functions.

Since $\sin \sqrt{x}$ is usually read "the sine of the square root of x," the function $x \to \sin \sqrt{x}$ is a composite of the sine and the square root functions. If we let $u = g(x) = \sqrt{x}$ and $h(u) = \sin u$, we have

$$\sin \sqrt{x} = h(g(x)).$$

Example 8-7b. Express $x \to x^{2/3}$ as the composite of two simpler functions in two ways.

The expression $x^{2/3}$ can be read as

(2) "the cube root of the square of x"

or

(3) "the square of the cube root of x."

Put $g(x) = x^2 = u$ and $h(u) = \sqrt[3]{u} = v$. In symbolic form (2) becomes

$$x^{2/3} = h(u) = h(g(x)), \tag{4}$$

while (3) becomes

$$x^{2/3} = g(v) = g(h(x)). \tag{5}$$

In other words, in this case, it doesn't matter whether we square first and then take the cube root, or take the cube root and then square. It should, however, be noted that generally the order of composition is important. In the Example 8-7a we had

$$\sin \sqrt{x} = h(g(x)), \quad \text{where} \quad g(x) = \sqrt{x} = u \quad \text{and} \quad h(u) = \sin u.$$

Reversing the order of composition, we have

$$g(h(x)) = \sqrt{\sin x},$$

which is certainly not the same as $\sin \sqrt{x}$.

It should be observed that there are other ways of expressing $x \to x^{2/3}$ as a composite. For example,

$$x^{2/3} = f(g(x)), \tag{6}$$

where $g(x) = (x - 1)^{1/3}$ and $f(x) = (x^3 + 1)^{2/3}$, since

$$f(g(x)) = [(x - 1) + 1]^{2/3} = x^{2/3}.$$

Exercises 8-7

1. Express each of the following as a composite of two functions which are polynomials, exponentials, logarithms, power, sine or cosine functions.

   (a) $x \to \sqrt{1 - x^2}$

   (b) $x \to e^{x^2}$

   (c) $x \to \cos (x^2 - 3x)$

   (d) $x \to \frac{1}{1 + x^2}$

   (e) $x \to \log_e \sqrt{x^2 + 1}$

   (f) $x \to (2 - 3x^2)^{100}$

   (g) $x \to (2x^2 - 2x + 1)^{-1/2}$

   (h) $x \to \log_e (\sin x)^2$

   (i) $x \to e^{\cos^2 x}$

   (j) $x \to 3e^{2 \sin x}$

   (k) $x \to 2^{(x+1)^2}$

2. Express each of the following as the composition of three or more simpler functions.

   (a) $x \to \log_e |8x^2 + 5x + 2|$

   (b) $x \to \sqrt{1 + \cos x}$

   (c) $x \to \cos(\sin(\cos x))$

   (d) $x \to (x + 1)^{3/5}$

   (e) $x \to \sqrt{1 - (\log_e x)^2}$

   (f) [illegible] $\to \frac{1}{1 + e^{2x}}$

3. Express $x \to |x|$ as a composite of the function $x \to x^2$ and some other function.

4. (a) Show that the composite of two linear functions is linear.

   (b) Exhibit the composite of two quadratic functions. What is the degree of this composition?

   (c) Is the composite of two polynomial functions a polynomial function? If so, what is its degree?

5. (a) If $u : x \to x$ and $f : x \to u(u(x))$ what is $f(3)$?

   (b) Suppose $u : x \to \frac{1}{x}$. Find an expression for f, the function defined by $f(x) = u(u(x))$.

6. (a) Show that composition of power functions is a commutative operation, that is, if $u : x \to x^a$ and $v : x \to x^b$ then $u(v(x)) = v(u(x))$.

   (b) Is the result of part (a) true for $u : x \to \cos x$ and $v : x \to \sin x$.

   (c) Is the result of part (a) true for exponential functions $u : x \to a^x$ and $v : x \to b^x$? $(a, b > 1)$

   (d) Is the result of part (a) true for $u : x \to e^x$ and $v : x \to \log_e x$?

7. Express the following as a composition of two functions

   (a) $x \to \int_{-2}^{x^2} t^{2/3}\,dt$

   (b) $x \to \int_{\sin x}^{1} e^t\,dt$

   (c) $x \to \int_{0}^{x^2} e^{-t^2}\,dt$

8. What is the domain of the function

$$x \to \sqrt{1 - (\log_e x)^2}?$$

8

8-8. The Chain Rule

Suppose we can express f as a composite of two functions g and h whose derivatives are known. The derivative of f can then be expressed in terms of the derivatives of g and h.

$$\boxed{\begin{array}{ll} \text{If} & f(x) = g(h(x)) \\ \text{then} & f'(x) = g'(h(x))h'(x). \end{array}} \tag{1}$$

This result is usually known as the chain rule. We have used the chain rule for particular functions in the case where h is a linear function. For example, suppose

$$f : x \to \sin(ax + b)$$

so that

$$f(x) = g(h(x))$$

where $g : u \to \sin u$ and $h : x \to ax + b = u$. Since $g' : u \to \cos u$ and $h' : x \to a$, the chain rule (1) gives

$$\begin{aligned} f'(x) &= g'(h(x))h'(x) \\ &= [\cos(ax + b)]a \\ &= a \cos(ax + b), \end{aligned}$$

which agrees with our previous result.

The general result for linear substitution is as follows. Suppose $f(x) = g(ax + b)$. Let $h(x) = ax + b$. The chain rule (1) gives

$$\begin{aligned} f'(x) &= g'(ax + b)h'(x) \\ &= ag'(ax + b) \end{aligned}$$

which shows that replacement of x by $ax + b$ in a general function g multiplies the derivative by a.

A special case of the chain rule was used in Section 6-7 to differentiate a power function. Suppose $f : x \to x^r$. We can write $f(x) = g(h(x))$, where $g : u \to e^u$ and $h : x \to r \log_e x = u$. The derivatives of g and h are given by

$$g' : u \to e^u \quad \text{and} \quad h' : x \to \frac{r}{x} .$$

The chain rule gives

$$\begin{aligned} f'(x) &= g'(h(x))h'(x) = g'(r \log_e x) \cdot \frac{r}{x} \\ &= e^{r \log_e x} \cdot \frac{r}{x} \\ &= x^r \cdot \frac{r}{x} \\ &= r\, x^{r-1}. \end{aligned}$$

Let us now prove the chain rule by generalizing the tangent approximation arguments used in Section 6-7. Suppose that f is related to g and h by composition

$$f(x) = g(h(x)).$$

If h is differentiable at a and g is differentiable at $h(a)$, we can write

(2) $$h(x) \approx h(a) + h'(a)(x - a), \text{ for } x \text{ close to } a,$$

and

(3) $$g(u) \approx g(h(a)) + g'(h(a))(u - h(a)), \text{ for } u \text{ close to } h(a).$$

In particular, if x is close to a the second term of (2) is close to zero so that $h(x)$ is close to $h(a)$.

We can replace u by $h(x)$ in (3) to obtain

$$g(h(x)) \approx g(h(a)) + g'(h(a))(h(x) - h(a)),$$

which will hold if x is close to a (so that $h(x) \approx h(a)$). We now use (2) again, this time to replace $h(x) - h(a)$ by $h'(a)(x - a)$. Thus, we have

(4) $$g(h(x)) \approx g(h(a)) + g'(h(a))h'(a)(x - a).$$

By assumption $f(x) = g(h(x))$ so we can rewrite (4) as

$$f(x) \approx f(a) + g'(h(a))h'(a)(x - a)$$

then subtract $f(a)$ and divide by $x - a$ to obtain

$$\frac{f(x) - f(a)}{x - a} \approx g'(h(a))h'(a).$$

Therefore,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = g'(h(a))h'(a),$$

which establishes the chain rule:

$$f'(a) = g'(h(a))h'(a).$$

The Leibniz notation $\frac{dy}{dx}$ for the derivative provides a convenient mnemonic device for the chain rule. Suppose $y = g(h(x))$; that is

$$y = g(u) \quad \text{where} \quad u = h(x).$$

We can then write $g'(u) = \frac{dy}{du}$, $h'(x) = \frac{du}{dx}$. The chain rule can then be expressed

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 8-8a. Find the derivative of $x \to \sqrt{x^2 + 1}$.

Put $g(x) = x^2 + 1 = u$ and $h(u) = \sqrt{u}$ so that

$$\sqrt{x^2 + 1} = h(g(x)).$$

Recall that $h'(u) = \frac{1}{2\sqrt{u}}$ and that $g'(x) = 2x$. The chain rule tells us that

$$\begin{aligned} D(\sqrt{x^2 + 1}) &= h'(g(x))g'(x) \\ &= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2 + 1}}. \end{aligned}$$

Example 8-8b. Find $D(e^{\sin x})$.

To express $x \to e^{\sin x}$ as a composite of functions with known derivatives, put

$$u = h(x) = \sin x, \quad g(u) = e^u$$

so that

$$e^{\sin x} = g(h(x))$$

and

$$h'(x) = \cos x, \quad g'(u) = e^u.$$

The chain rule gives

$$\begin{aligned} D(e^{\sin x}) &= g'(h(x)) \cdot h'(x) \\ &= e^{\sin x} \cdot \cos x. \end{aligned}$$

Example 8-8c. For $f : x \to (x^2 + x + 1)^{100}$, find $f'(-1)$.

We could expand and then differentiate. Obviously, such a procedure would be quite lengthy. Instead we let $h(x) = x^2 + x + 1 = u$ and $g(u) = u^{100}$, so that

$$f(x) = g(h(x)).$$

We have $h'(x) = 2x + 1$, $g'(u) = 100u^{99}$, so that (by the chain rule)

$$f'(x) = 100(x^2 + x + 1)^{99} \cdot (2x + 1).$$

Thus $f'(-1) = -100$.

Example 8-8d. Use the chain rule to show that $D(\log_e (\cos x)) = -\tan x$, thus verifying integration formula 12 of the Table of Integrals:

$$\int \tan x \, dx = -\log_e (\cos x).$$

Put $h(x) = u = \cos x$, $g(u) = \log_e u$, so that

$$\log_e (\cos x) = g(h(x)),$$

and hence

$$\begin{aligned} D(\log_e (\cos x)) &= g'(h(x))h'(x) \\ &= \frac{1}{h(x)} \cdot (-\sin x) \\ &= -\frac{\sin x}{\cos x} \\ &= -\tan x. \end{aligned}$$

Example 8-8e. Find $\frac{dy}{dx}$ if $y = \frac{1}{1 + \sin (x^2)}$.

We let $u = x^2$ and $v = 1 + \sin u$, whence $y = \frac{1}{1 + \sin u} = \frac{1}{v}$. We obtain $\frac{du}{dx} = 2x$, $\frac{dv}{du} = \cos u$, and $\frac{dy}{dv} = -\frac{1}{v^2}$. We have $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$. Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \left(-\frac{1}{v^2}\right)(\cos u)(2x)\\ &= \left(-\frac{1}{(1+\sin u)^2}\right)(\cos u)(2x)\\ &= -\frac{2x\cos(x^2)}{(1+\sin(x^2))^2}\ .\end{aligned}$$

Example 8-8f. Analyze the graph of $y = xe^{-x^2}$.

The product rule gives

$$\begin{aligned}y' = D\left(xe^{-x^2}\right) &= xD\left(e^{-x^2}\right) + e^{-x^2}\,Dx\\ &= xD\left(e^{-x^2}\right) + e^{-x^2}.\end{aligned}$$

Applying the chain rule to e^{-x^2}, we get

(5) $$D\,e^{-x^2} = e^{-x^2}(-2x) = -2xe^{-x^2},$$

so that

(6) $$\begin{aligned}y' &= -2x^2\,e^{-x^2} + e^{-x^2}\\ &= (-2x^2+1)e^{-x^2}.\end{aligned}$$

We note that y' will have the same sign as

$$-2x^2 + 1 = -2\left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right).$$

The graph falls until it reaches $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}e^{-1/2}\right)$, then rises to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{-1/2}\right)$, then falls.

To analyze convexity we find the second derivative. Apply the product rule to (6) to obtain

$$\begin{aligned}y'' &= D\left[(-2x^2+1)e^{-x^2}\right]\\ &= (-2x^2+1)D\left(e^{-x^2}\right) + e^{-x^2}\,D(-2x^2+1).\end{aligned}$$

Now use (5) and the fact that $D(-2x^2+1) = -4x$ to obtain

$$y'' = (-2x^2 + 1)\left(-2xe^{-x^2}\right) + e^{-x^2}(-4x)$$
$$= (4x^3 - 6x)e^{-x^2} .$$

The second derivative y'' has the same sign as

$$4x^3 - 6x = 4x(x - \sqrt{\tfrac{3}{2}})(x + \sqrt{\tfrac{3}{2}}) .$$

The graph is convex for $-\sqrt{\frac{3}{2}} < x < 0$ or $\sqrt{\frac{3}{2}} < x$, and concave for $x < -\sqrt{\frac{3}{2}}$ or $0 < x < \sqrt{\frac{3}{2}}$. We can show that if $|x|$ is large then $xe^{-x^2} \approx 0$, so that the x-axis is an asymptote. We know that $|x|e^{-|x|}$ approaches 0 if $|x|$ is large. Then noting that $-x^2 \leq -|x|$ if $|x| \geq 1$, we have $e^{-x^2} \leq e^{-|x|}$, since $x \to e^x$ is an increasing function. Therefore, we have

$$|xe^{-x^2}| \leq |x| \, e^{-|x|} \approx 0, \text{ if } |x| \text{ is large.}$$

See Figure 8-8a for the graph of $y = xe^{-x^2}$.

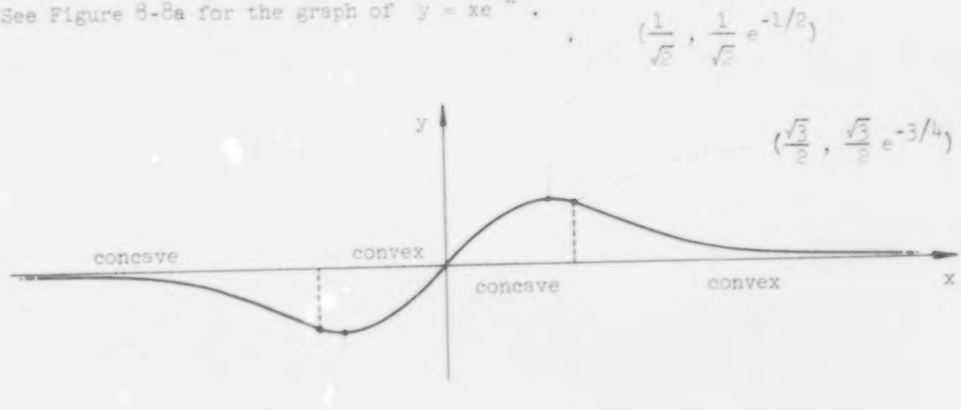


Figure 8-8a

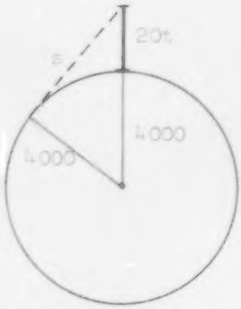
$y = xe^{-x^2}$

Related Rates

In Section 2-8 we discussed the distance, velocity, speed, and acceleration of a particle moving in a straight line. The distance traveled s depends on the time t, according to some law which defines a function $f : t \rightarrow s = f(t)$. We have thought of the velocity $v = \frac{ds}{dt} = f'(t)$ as the rate of change of distance with respect to time.

As we know from Section 4-4, we are not limited to particles moving in a straight line. Furthermore, we can consider their relative motions, as we did in Section 4-4, with point Q moving along the x-axis as point P moved around a circle.

Example 8-8g. If a helicopter rises vertically from the surface of the earth at the constant speed of 20 mi./hr., how fast is its line-of-sight to the horizon increasing after 6 minutes? (Assume that the earth is a perfect sphere with 4000 mile radius.)



Since the line-of-sight is tangent to the earth at the horizon, it is perpendicular to the radius of the earth there. At time t (in hours) the height of the helicopter is $20t$ (miles), and so by the Pythagorean Theorem,

$$s = \sqrt{(20t + 4000)^2 - 4000^2},$$

where s represents the length of the line-of-sight to the horizon. Differentiating with respect to time,

$$\begin{aligned}\frac{ds}{dt} &= \frac{1}{2}[(20t + 4000)^2 - 4000^2]^{-1/2}[2(20t + 4000)(20)] \\ &= \frac{20(20t + 4000)}{[(20t + 4000)^2 - 4000^2]^{1/2}} .\end{aligned}$$

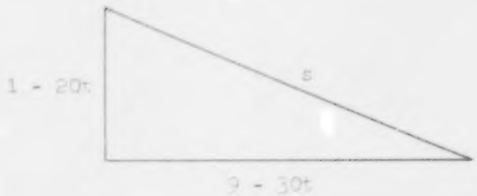
After 6 minutes, $t = \frac{1}{10}$ (hours), and

$$\left.\frac{ds}{dt}\right|_{t=\frac{1}{10}} = \frac{20(2 + 4000)}{[(2 + 4000)^2 - 4000^2]^{1/2}} = \frac{80040}{[16004]^{1/2}} \approx \frac{80040}{126.5} .$$

Therefore, the line-of-sight is increasing at a rate of approximately 633 mi./hr. [illegible] 10.5 mi./min.

Example 8-8h. For the last 3 minutes of its flight prior to splashdown, the moonship Columbia descended at an average rate of 20 mi./hr., approximately. The aircraft carrier Hornet was steaming directly toward the point of splashdown at the constant rate of 30 mi./hr. If the carrier was 9 miles from the point of splashdown at 9:47 a.m. PDT July 25, 1969, how fast was the distance between the carrier and the Columbia decreasing at 9:49 a.m., 1 minute before splashdown?

Let t represent the time elapsed after the point 3 minutes prior to splashdown. If t is measured in hours, the distance Columbia falls is $20t$ and the distance traveled by the carrier is $30t$. At $t = 0$ the Columbia is at an altitude of 1 mile (the distance it falls in 3 minutes), and the carrier is 9 miles away from the point of splashdown, so at time t.



Columbia is $(1 - 20t)$ miles above the point of splashdown and the carrier is $(9 - 30t)$ miles away. The distance between them at time t is

$$s = \sqrt{(1 - 20t)^2 + (9 - 30t)^2}$$

Hence,

$$\frac{ds}{dt} = \frac{1}{2}[(1 - 20t)^2 + (9 - 30t)^2]^{-1/2}[2(1 - 20t)(-20) + 2(9 - 30t)(-30)]$$

$$= \frac{-20(1 - 20t) - 30(9 - 30t)}{[(1 - 20t)^2 + (9 - 30t)^2]^{1/2}} .$$

One minute before splashdown $t = \frac{1}{30}$, so

$$\left.\frac{ds}{dt}\right|_{t=\frac{1}{30}} = \frac{-20(1 - \frac{20}{30}) - 30(9 - \frac{30}{30})}{[(1 - \frac{20}{30})^2 + (9 - \frac{30}{30})^2]^{1/2}}$$

$$= \frac{-\frac{20}{3} - 30(8)}{[\frac{1}{9} + 64]^{1/2}} \approx \frac{-246.6}{8}$$

$$\approx -30.8.$$

Hence, the distance between the Columbia and the carrier is decreasing at the approximate rate of 30.8 mi./hr. at 9:49 a.m., one minute before splashdown.

Exercises 8-8

1. Find the derivatives of each of the following by making an appropriate substitution:

(a) $x \to \sqrt{1 - x^8}$

(b) $x \to e^{x^2}$

(c) $x \to \cos (x^3 - 3x)$

(d) $x \to \frac{1}{1 + x^2}$

(e) $x \to \log_e \sqrt{x^2 + 1}$

(f) $x \to (2 - 3x^2)^{100}$

(g) $x \to (2x^2 - 2x + 1)^{-1/2}$

(h) $x \to \log_e (\sin x)^2$

(i) $x \to e^{\cos^2 x}$

(j) $x \to 3e^{2 \sin x}$

(k) $x \to 2^{(x+1)^2}$

2. Find the derivatives of each of the following functions by making one or more substitutions.

(a) $x \to \sqrt{1 + \cos x}$

(b) $x \to \sqrt{1 - (\log_e x)^2}$

(c) $x \to \frac{1}{1 + e^{2x}}$

(d) $x \to \cos(\sin(\cos x))$

3. Find the derivatives of each of the following functions by using the chain rule, along with the sum and product rules.

(a) $x \to (x^2 + 1)^{1/2} + (x^2 + 1)^{-1/2}$

(b) $x \to \frac{\sqrt{x^2 - a^2}}{\sqrt{x^2 + a^2}} = [x^2 - a^2]^{1/2}[x^2 + a^2]^{-1/2}$

(c) $x \to x(2x^2 + 2x + 1)^{-1/2}$

(d) $x \to x^2 \sqrt{\sin x}$

(e) $x \to \sin^2(e^x)$

(f) $x \to e^{x \sin x}$

(g) $x \to \log_e (\sqrt{x} \cos x)$

(h) $x \to e^{\log_e x + \cos x}$

(i) $x \to \sin x \cos x \log_e \sqrt{x}$

(j) $x \to \cos^2 (\log_e x) + \sin^2 (\log_e x)$

4. (a) Show that if $f(x) = \int_a^{g(x)} h(t)dt$ then $f'(x) = h(g(x))g'(x)$.

   (b) Deduce from (a) that if $F(x) = \int_{x^2}^{b} f$ then $F'(x) = -2x\, f(x^2)$.

   (c) Verify (a) by evaluating $\int_{-\pi}^{x^2} \sin t\, dt$ and then calculating its derivative.

5. Find the derivatives of each of the following functions

   (a) $x \to \int_{-2}^{x^2} t^{2/3}\, dt$

   (b) $x \to \int_{\sin x}^{1} e^t\, dt$

   (c) $x \to \int_{0}^{x^2} e^{-t^2}\, dt$

6. (a) Find the derivative of $f : x \to x^x$, $x > 0$. (Hint: Write $x^x = e^{x \log_e x}$.)

   (b) What is the minimum value of f.

   (c) Find the second derivative of f and show that the graph of f is convex.

7. Determine intervals of increase-decrease and convexity-concavity. Then sketch a graph.

   (a) $f : x \to \dfrac{x}{x^2 - 1} = [x(x^2 - 1)^{-1}]$

   (b) $f : x \to e^{1/x}$

   (c) $f : x \to \log_e \dfrac{1 + x^2}{1 - x^2}$, $-1 < x < 1$

8. Find the equation of the tangent line to the curve at the point indicated:

   (a) $y = xe^{-x^2}$, $x = 0$

   (b) $y = e^{-11x^2}$, $x = 1$

   (c) $y = \sin(\pi - x^2)^{3/2}$, $x = \sqrt{\pi}$

   (d) $y = \log_e (1 - x^2)$, $x = \frac{1}{2}$

   (e) $y = e^{e^x}$, $x < 0$

   (f) $y = (e^x)^\pi$, $x = e$

9. If $f(x) = (Ax + B)\sin x + (Cx + D)\cos x$, determine the value of constants A, B, C, D such that for all x, $f'(x) = x \sin x$.

10. If $g(x) = (Ax^2 + Bx + C)\sin x + (Dx^2 + Ex + F)\cos x$, determine the value of constants A, B, C, D, E, F such that for all x, $g'(x) = x^2 \cos x$.

The notation $\left.\frac{dy}{dx}\right|_{x=a}$ is sometimes used for the value of the derivative of y at $x = a$. This notation is used in the following problems.

11. Let $y = \sin x$ and $x = t^2 + \frac{1}{t}$. Find $\left.\frac{dy}{dt}\right|_{t=1}$ and $\left.\frac{dy}{dx}\right|_{x=2}$.

12. Let $y = f(x)$ and $x = h(t)$. Express $\left.\frac{dy}{dt}\right|_{t=t_0}$ in terms of t_0.

13. Let $y = f(x)$, $x = h(t)$, $x_0 = h(t_0)$. Show that

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}} .$$

14. Find the following:

   (a) $D \sin x\big|_{x=0} + D \sin x\big|_{x=\pi/4}$

   (b) $D(x^2 + \sin a \sin x)\big|_{x=5\pi/3}$

(c) $\frac{d}{dx}(x^2 - a^2)\Big|_{x=a}$ $[\frac{d}{dx} = D]$

(d) $D(f(a)\sin x + f(x)\sin a + f(x)\sin x)\Big|_{x=a}$

15. A spherical balloon if being filled with helium at the rate of 100 in.^{3}/min. How fast is the radius increasing when it has reached the value of 5 inches?

16. A car crosses a railroad track moving perpendicular to the track at the rate of 40 mi./hr. One quarter hour later a train crosses the same intersection moving 72 mi./hr. along the track. How fast are the car and train separating one hour after the car passed the intersection?

17. A small rocket is shot straight up from a point 75 feet away from an observer. If the rocket travels at the constant rate of 100 ft./sec., how rapidly will it be receding from the observer 3 seconds later?

8-9. The General Power and Reciprocal Rules

A special case of the chain rule, known as the general power rule, occurs so frequently that it is worth discussing separately.

Suppose the values of the function f can be expressed as

$$f(x) = (h(x))^r$$

where r is a fixed real number and h is a function. In other words,

$$f(x) = g(h(x)), \text{ where } h : x \to h(x) = u \text{ and } g : u \to u^r.$$

If h is differentiable at x and if $r(h(x))^{r-1}$ is defined (that is, if g is differentiable at u), then the chain rule gives

$$f'(x) = g'(h(x))h'(x).$$

Since $g' : u \to ru^{r-1}$, we can write this as

$$(1) \qquad f'(x) = r(h(x))^{r-1}h'(x).$$

This is the general power rule. Using the D notation it can be expressed as

$$(2) \qquad \boxed{Du^r = ru^{r-1}\,Du.}$$

For example, suppose

$$f : x \to \sin^3 x$$

that is

$$f(x) = (h(x))^3, \text{ where } h : x \to \sin x.$$

The power formula (1) gives

$$\begin{aligned} f'(x) &= 3(h(x))^2 h'(x) \\ &= 3 \sin^2 x \cos x. \end{aligned}$$

As an example of the case when the exponent r is not an integer, consider the function

$$f : x \to \sqrt{x^2 + 1} = (x^2 + 1)^{1/2}.$$

The power formula gives

$$\begin{aligned} f'(x) &= D[(x^2+1)^{1/2}] = \tfrac{1}{2}(x^2+1)^{-1/2}\,D(x^2+1) \\ &= \tfrac{1}{2}(x^2+1)^{-1/2}\cdot 2x \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

As an example of the case when r is a negative integer, consider the function

$$f : x \to \frac{1}{(\log_e x)^2} = (\log_e x)^{-2}.$$

The power formula then gives

[illegible] $$\begin{aligned}) &= D[(\log_e x)^{-2}] = -2(\log_e x)^{-3}\,D(\log_e x) \\ &= -2(\log_e x)^{-3}\cdot\frac{1}{x} \\ &= \frac{-2}{x(\log_e x)^3} \end{aligned}$$

The case when $r = -1$ is so important that it deserves special consideration. Suppose the values of the function f can be expressed as

$$f(x) = \frac{1}{g(x)},$$

where g is a function. We can then write

$$f(x) = (g(x))^{-1}$$

and apply the power formula to obtain

$$\begin{aligned} f'(x) &= D[(g(x))^{-1}] = -(g(x))^{-2}\,D(g(x)) \\ &= -(g(x))^{-2}\,g'(x) \\ &= \frac{-g'(x)}{(g(x))^2}. \end{aligned}$$

This will hold, provided $g(x) \neq 0$ and g is differentiable at x. In words, the derivative of the reciprocal of a function is the negative of the derivative of the function times the reciprocal of the square of the function. Using D notation, we summarize:

$$\boxed{D\left(\frac{1}{g(x)}\right) = \frac{-D\,g(x)}{[g(x)]^2}}$$

We shall refer to this as the reciprocal rule.

For example, suppose

$$f : x \to \frac{1}{x^2 + 2} .$$

The reciprocal rule gives

$$f'(x) = D\left(\frac{1}{x^2 + 2}\right) = -\frac{D(x^2 + 2)}{(x^2 + 2)^2}$$

$$= \frac{-2x}{(x^2 + 2)^2} .$$

A differentiation formula for the secant function can be found using the reciprocal rule. The secant function is defined by

$$\sec : x \to \frac{1}{\cos x} .$$

The expression $\frac{1}{\cos x}$ is not defined if $\cos x = 0$, that is, if x is an odd multiple of $\frac{\pi}{2}$. Thus the secant function is defined only for those values x which are not odd multiples of $\frac{\pi}{2}$. The reciprocal rule gives the derivative

$$D(\sec x) = D\left(\frac{1}{\cos x}\right) = -\frac{D(\cos x)}{\cos^2 x}$$

$$= -\frac{(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} .$$

Since $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ this result is usually expressed as

$$\boxed{D(\sec x) = \sec x \tan x.} \tag{4}$$

A corresponding formula for the cosecant function is given in the exercises.

Exercises 8-9

1. **Use the power formula to find the derivative of each of the following:**

(a) $x \to \sqrt{\sin x}$

(b) $x \to (\log_e x)^{\pi}$

(c) $s \to (s^3 + 3s)^{25}$

(d) $t \to (e^t)^{-10}$

(e) $x \to \dfrac{1}{\sqrt[3]{(1 - x)^2}}$

(f) $t \to (1 + \frac{1}{t})^{4/3}$

(g) $v \to \cos^{10} 2v$

(h) $x \to \left(\displaystyle\int_0^x \sqrt{t^3 + 1}\, dt\right)^{1/2}$

2. Use the reciprocal rule to find $\dfrac{dy}{dx}$ if

(a) $y = \dfrac{1}{1 - x^2}$

(b) $y = \left(\dfrac{1}{1 - x^2}\right)^{-5}$

(c) $y = \dfrac{1}{1 + e^{2x}}$

(d) $y = (1 + \log_e x)^{-1}$

(e) $y = \dfrac{1}{\sqrt{x + \frac{1}{x}}}$

(f) $y = (\sin x + \cos x)^{-1}$

3. Find an equation for the tangent line to each of the following curves at the indicated point.

(a) $y = \sin^{3/2} (2x), \quad x = \frac{\pi}{6}$

(b) $y = \left(\displaystyle\int_0^x e^{-t^2}\, dt\right)^2, \quad x = 0$

(c) $s = \sqrt{t + \frac{1}{t}}, \quad t = 1$

4. For each of the following

(i) state where defined,
(ii) find the intervals of increase-decrease,
(iii) convexity-concavity,
(iv) asymptotes (if any), and
(v) sketch.

(a) $y = \dfrac{1}{1 + x^2}$

(b) $y = \sqrt{\sin x}$

5. Show that each of the following is an increasing function

   (a) $x \to \frac{1}{1 - e^x}$, $x > 0$

   (b) $x \to (x^3 + 3x)^{10}$, $x \geq 0$

6. Find expressions for the derivatives if

   (a) $y = \sec x = \frac{1}{\cos x}$

   (b) $y = \csc x = \frac{1}{\sin x}$

   (c) $y = \tan x = \frac{\sin x}{\cos x} = (\sin x)(\cos x)^{-1}$

   (d) $y = \cot x = \frac{\cos x}{\sin x}$

   Use the results of (a), (b), (c) and (d) to obtain the following:

   (e) $D(\tan 3x)$

   (f) $D\sqrt{\tan 2x}$

   (g) $D(\sec^2 x^2)$

   (h) $D(\csc 3x)^{1/6}$

   (i) $D[\sec(\csc x)]$

7. In what intervals is the secant function increasing? convex? Sketch its graph.

8. (a) Find $D(\sec x \csc x)$

   (i) in terms of $\sec x$ and $\csc x$
   (ii) in terms of $\tan x$ and $\cot x$
   (iii) in terms of $\csc 2x$ and $\cot 2x$

   (b) Find

   (i) $D(\tan x \cot x)$
   (ii) $D(\sin x \csc x)$
   (iii) $D(\cos x \sec x)$

   (c) Find

   (i) $D(\sin x \cot x)$
   (ii) $D(\cos x \tan x)$

9. Show that

(a) $D\left(\frac{\tan^{(k+1)} x}{k+1}\right) = \tan^k x \sec^2 x, \quad k \neq -1$

(b) $D(\frac{1}{k} \csc^k x) = -\csc^k x \cot x, \quad k \neq 0$

(c) $D(\cot^2 x) = D(\csc^2 x)$

10. (a) Use the product and reciprocal rules to show that $(\frac{u}{v})' = \frac{uv' - u'v}{v^2}$.

(b) Find $D\left(\frac{x^2 + 1}{3x^2 - x}\right)$

8-10. The Quotient Rule

By combining the product rule and the reciprocal rule we can obtain a rule for differentiating quotients of functions. Suppose the values of the function f can be expressed as

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are functions (and, of course, $q(x) \neq 0$). It is then common to write $f = \frac{p}{q}$ and call f the quotient of p and q. Since we can write

$$f(x) = p(x) \cdot \frac{1}{q(x)},$$

the function f is just the product of p and the reciprocal of q. If p and q are differentiable at x and $q(x) \neq 0$, then the product rule gives

$$\begin{aligned} f'(x) &= D(p(x) \cdot \frac{1}{q(x)}) \\ &= p(x)\ D(\frac{1}{q(x)}) + \frac{1}{q(x)}\ D\ p(x). \end{aligned}$$

The reciprocal rule gives

$$D(\frac{1}{q(x)}) = \frac{-q'(x)}{(q(x))^2},$$

so that

$$\begin{aligned} f'(x) &= p(x)\left(\frac{-q'(x)}{(q(x))^2}\right) + \frac{1}{q(x)}\ p'(x) \\ &= \frac{-p(x)q'(x) + q(x)p'(x)}{(q(x))^2}. \end{aligned}$$

This is usually written in the form

(1) $$f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$$

and is referred to as the quotient rule. With D notation it can be written as

(2) $$\boxed{D(\frac{p(x)}{q(x)}) = \frac{q(x)\ Dp(x) - p(x)\ Dq(x)}{(q(x))^2}}$$

In words, the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all over the square of the denominator.

Example 8-10a. Use the quotient rule to find the derivative of the tangent function and discuss its graph in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The tangent function can be expressed as

$$\tan : x \rightarrow \frac{\sin x}{\cos x} .$$

This function is defined for those x for which $\cos x \neq 0$; that is, the tangent function is defined only when x is not an odd multiple of $\frac{\pi}{2}$. The quotient rule gives the derivative

$$D(\tan x) = D\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\, D(\sin x) - \sin x\, D(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} .$$

Since $\sec x = \frac{1}{\cos x}$ this is usually expressed as

$$(3) \qquad \boxed{D(\tan x) = \sec^2 x.}$$

The function $x \rightarrow \cos x$ is not zero in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ so that

$$\sec^2 x > 0 \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2} .$$

Therefore, the tangent function is an increasing function in the interval. In fact the tangent function is strictly increasing on this interval.

Let us denote the second derivative of $y = \tan x$ by y''. We have

$$y'' = D(\sec^2 x) = 2 \sec x\, D(\sec x)$$

$$= 2 \sec x\, (\sec x \tan x)$$

$$= 2 \sec^2 x \tan x,$$

where we used the power rule and the fact (Section 8-9, (4)) that $D(\sec x) = \sec x \tan x$.

The second derivative $y'' = 2\sec^2 x \tan x$ will be negative for $-\frac{\pi}{2} < x < 0$ and positive for $0 < x < \frac{\pi}{2}$; that is, the graph of the tangent function is concave in the left interval and convex in the right interval.

As x approaches $\frac{\pi}{2}$, $\cos x$ approaches zero while $\sin x$ approaches 1. Thus the line given by $x = \frac{\pi}{2}$ is an asymptote and $y = \tan x$ becomes large without bound as x approaches $\frac{\pi}{2}$ from the left; similarly we could argue that $y = \tan x$ grows large without bound through negative values as x approaches $-\frac{\pi}{2}$ from the right. A graph of the tangent function in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is given in Figure 8-10a.

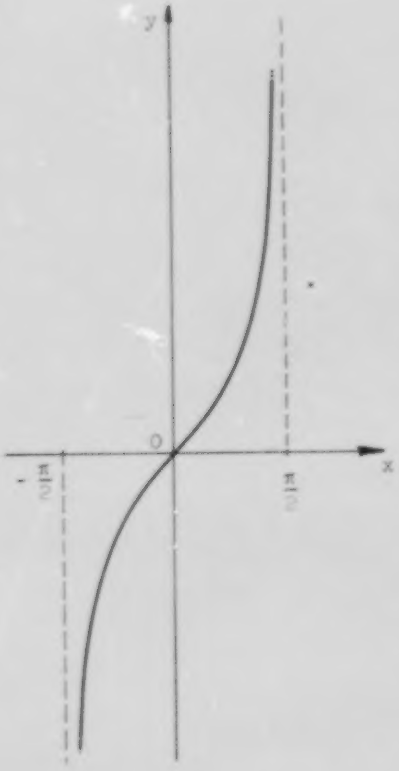


Figure 8-10a

$y = \tan x$

Rational functions, that is, quotients of polynomials, can be differentiated using the quotient rule. Such a function is discussed in the following example.

Example 8-10b. Disucss the graph of the function

$$f : x \to \frac{x^3 + x^2 - 1}{x^2 - 1} .$$

This function is not defined when $x = \pm 1$. As x approaches $+1$ from the left the numerator approaches 1, while the denominator is negative and near zero. Thus $|f(x)|$ becomes large and $f(x)$ negative as x approaches $+1$ from the left; that is, $f(x)$ grows large without bound through negative values. Similar arguments show that $f(x)$ grows large without bound through positive values as x approaches $+1$ from the right.

Suppose x approaches -1 from the left. The numerator approaches -1, while the denominator is positive and approaches 0. Thus as x approaches -1 from the left, $f(x)$ grows large without bound through negative values.

To discuss the behavior when $|x|$ is large we rewrite the expression for $f(x)$ as

$$x\left(\frac{1 + \frac{1}{x} - \frac{1}{x^3}}{1 - \frac{1}{x^2}}\right).$$

If $|x|$ is large, the expression in the parenthesis is nearly 1. Thus $f(x)$ behaves like x for large values, positive or negative.

Note that f is continuous except when $x = \pm 1$. For example, if $a \neq \pm 1$ then as x approaches a, the numerator approaches $a^3 + a^2 - 1$, while the denominator approaches $a^2 - 1$. Thus $f(x)$ approaches $\frac{a^3 + a^2 - 1}{a^2 - 1} = f(a)$. This is illustrative of the fact that a rational function is continuous except at the zeros of its denominator.

We now determine the intervals of increase and decrease. The quotient rule gives:

$$f'(x) = \frac{(x^2 - 1)D(x^3 + x^2 - 1) - (x^3 + x^2 - 1)D(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)(3x^2 + 2x) - (x^3 + x^2 - 1)(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 3x^2}{(x^2 - 1)^2} .$$

The derivative f' is a rational function. (In fact, the derivative of a rational function is always a rational function.) In factored form, we have

$$f'(x) = \frac{x^2(x - \sqrt{3})(x + \sqrt{3})}{(x - 1)^2(x + 1)^2} ,$$

from which we see that the sign of f' is determined by the sign of $(x - \sqrt{3})(x + \sqrt{3})$. We conclude that the graph of f is rising when $x < -\sqrt{3}$ or $x > \sqrt{3}$ and is falling when $-\sqrt{3} < x < -1$, $-1 < x < 0$, $0 < x < 1$ or $1 < x < \sqrt{3}$.

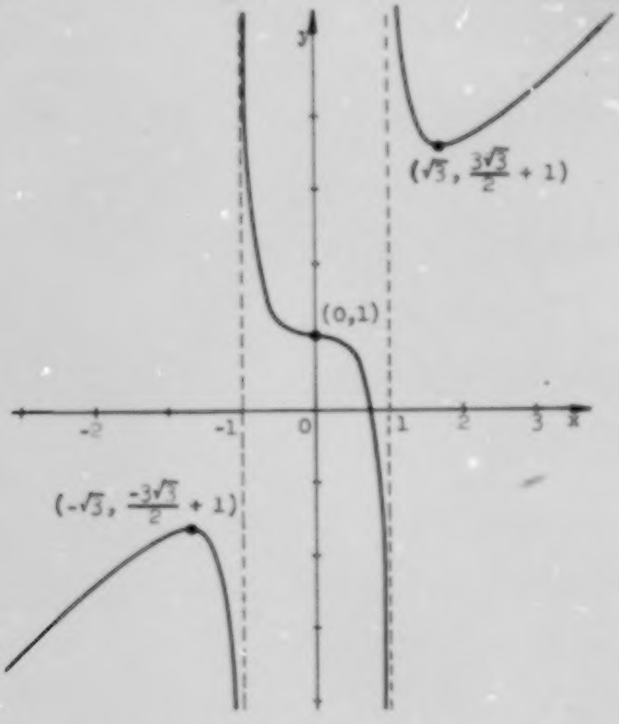


Figure 8-10b

Graph of $y = \frac{x^3 + x^2 - 1}{x^2 - 1}$

Exercises 8-10

1. Evaluate:

(a) $D(\frac{x}{x - 1})$

(b) $D\left(\frac{x^2}{1 + x^2}\right)$

(c) $D(1 - \frac{1}{x})^{-1}$

(d) $D\left(\frac{3 + 2x^2}{2 - x^2}\right)$

(e) $D(\frac{1}{x} + \frac{1}{1 - x})$

(f) $D\left(\frac{\sqrt{x}}{1 + x^2}\right)$

(g) $D\left(\frac{1}{1 + \sqrt{x}}\right)$

(h) $D\left(\frac{x^2 - 1}{x^2 + 1}\right)^{-1}$

(i) $D(\frac{\sin x}{1 + \tan x})$

(j) $D\left(\frac{e^x}{1 + x^2}\right)$

(k) $D\left(\frac{x \log_e x}{1 - 2x}\right)$

(ℓ) $D(\cos x \sec x)$

(m) $D\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

(n) $D[(1 + \frac{1}{x})(1 + \log_e x)]$

(o) $D\left(\frac{\log_e x^2}{\sqrt{x^2 + 1}}\right)$

2. Show that $D(\cot x) = -\csc^2 x$.

3. Discuss the graphs of each of the following, as in Example 8-10a, b. Sketch.

(a) $y = \frac{x + 2}{x^2 - 1}$

(b) $y = \frac{x - 1}{x + 1}$

(c) $y = \frac{e^{-2x}}{1 + x}$

4. Find

(a) $\int_0^{\pi/4} \sec^2 x \, dx$

(b) $\int_{-\pi/3}^{0} \sec x \tan x \, dx$

5. Let $y = f(t)$, $w = g(t)$, $t = h(x)$, $z = \frac{y}{w}$.

(a) Using Leibnizian notation, find $\frac{dz}{dx}$ in terms of $\frac{dy}{dt}$, $\frac{dw}{dt}$, and $\frac{dt}{dx}$.

(b) Using (a) express $\left.\frac{dz}{dx}\right|_{x=x_0}$ in terms of f', g', and h'.

8-11. Inverse Functions

Let us review our discussions of Section 5-1 and 6-1 where we defined the square root function and found its derivative. The function

$$g : x \rightarrow x^2, \; x \geq 0$$

is a strictly increasing function and its graph meets each horizontal line given by $y = c$, $c \geq 0$. In other words

$$g(x_1) < g(x_2) \quad \text{if} \quad 0 \leq x_1 < x_2$$

and each nonnegative number c is in the range of g; i.e., $c = g(d)$. The function

$$f : x \rightarrow \sqrt{x}$$

is defined for each nonnegative real number c by

$$f(c) = d \quad \text{if} \quad g(d) = c;$$

that is $\sqrt{c}$ is the nonnegative real number d such that $c = d^2$. This defines a function f, since for each $c \geq 0$ there is a unique $d \geq 0$ such that $c = d^2$. This follows from the fact that g is strictly increasing.

The graph of f is obtained by folding the graph of g over the line given by $y = x$; that is

(1) (c,d) lies on the graph of f if and only if (d,c) lies on the graph of g. (See Figure 8-11a.)

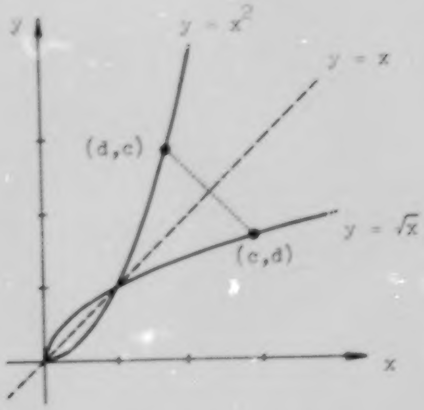


Figure 8-11a

The tangent to the graph of g at (d,c) is given by the equation

$$y = g(d) + g'(d)(x - d) = d^2 + 2d(x - d).$$

If $c > 0$ then d must also be positive and this line folds over the line given by $y = x$ into the line whose equation is

$$y = d + \frac{1}{2d}(x - d^2).$$

This is the tangent to the graph of f at the point (c,d). Replacing d by $\sqrt{c}$, we see that the tangent to the graph of f at (c,d) has the equation

$$y = \sqrt{c} + \frac{1}{2\sqrt{c}}(x - c).$$

The coefficient of x is the derivative of f at c, so that

$$(2) \qquad f'(c) = \frac{1}{2\sqrt{c}}, \ c > 0.$$

This same method was used to define

$$f : x \to \log_e x, \ x > 0$$

in terms of the function $g : x \to e^x$ and to obtain the derivative formula

$$f' : x \to \frac{1}{x}.$$

In this section we discuss a general form of the folding process. Suppose the function g is defined for those numbers x in an interval I, which may be the entire real number line (as in the case $g : x \to e^x$), a ray (as in the case $g : x \to x^2, x \geq 0$), or a line segment. Suppose further that g is continuous at each point of I and that g is strictly increasing; that is,

$$(3) \qquad g(x_1) < g(x_2) \ \text{if} \ x_1 \ \text{and} \ x_2 \ \text{are in} \ I \ \text{and} \ x_1 < x_2.$$

If we fold the graph of g over the line given by $y = x$, then we obtain the graph of a function f. The function is called the inverse of g and is defined by

$$f(c) = d \ \text{if} \ g(d) = c;$$

that is, $f(c)$ is defined for those numbers c in the range of g (meaning that $c = g(d)$ for some d in I). This defines a function since for a number c in the domain of f there is exactly one number d in I such that $g(d) = c$. This follows from the assumption (3) that g is strictly increasing. That the domain of f is an interval is a consequence of the assumption that

g is continuous. In the appendices, it will be shown that the inverse f is continuous at each point of its domain.

The graphs of f and g are related by the condition

(4) (c,d) lies on the graph of f if and only if (d,c) lies on the graph of g;

that is, the graph of the inverse f can be obtained by folding the graph of g over the line given by $y = x$.

The **folding** process used to find the derivative of the square root function also works in the general case. Suppose f is the inverse of the continuous function g and that $g(d) > 0$. The tangent to the graph of g at (d,c) has the equation

$$y = g(d) + g'(d)(x - d).$$

This folds over the line given by $y = x$ into the line whose equation is

$$y = d + \frac{1}{g'(d)}(x - c),$$

the equation of the tangent line to the graph of the inverse f at the point (c,d). The value $f'(c)$ is the coefficient of x,

$$f'(c) = \frac{1}{g'(d)}, \quad \text{if } g'(d) > 0.$$

To obtain a formula for $f'(c)$ in terms of c, we replace d by $f(c)$, to obtain the inverse function rule:

$$f'(c) = \frac{1}{g'(f(c))}, \quad \text{if } g'(f(c)) > 0. \tag{5}$$

The geometrically intuitive folding process can be justified by rigorous arguments. In the appendices it is shown that limit concepts give the same results; that is,

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

is indeed equal to $\dfrac{1}{g'(f(c))}$.

Definitions and derivatives of the inverse circular functions can be obtained using this process.

The Arcsine Function

If we restrict the sine function to an interval in which it is strictly increasing then the methods we have been using can be applied to obtain an inverse function. It is conventional to use the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The function $g : x \to \sin x$ is strictly increasing on this interval. Its inverse function f is usually called the arcsine (or inverse sine) function, and denoted by arcsin. The range of g is the interval $-1 \leq x \leq 1$ so that

$$f : x \to \arcsin x$$

is defined for $-1 \leq x \leq 1$. Its value at c, $\arcsin c$, is that real number d, such that

$$\sin d = c \text{ and } -\frac{\pi}{2} \leq d \leq \frac{\pi}{2}.$$

In other words,

(6) $$f(c) = d \text{ if and only if } |d| \leq \frac{\pi}{2} \text{ and } \sin d = c.$$

For example,

$$\sin 0 = 0; \sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}; \sin \frac{\pi}{2} = 1$$

so that

$$\arcsin 0 = 0; \arcsin(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}; \arcsin 1 = \frac{\pi}{2}.$$

The graph of $f : x \to \arcsin x$ can be obtained by folding the graph of $g : x \to \sin x$ over the line given by $y = x$, as shown in Figure 8-11b.

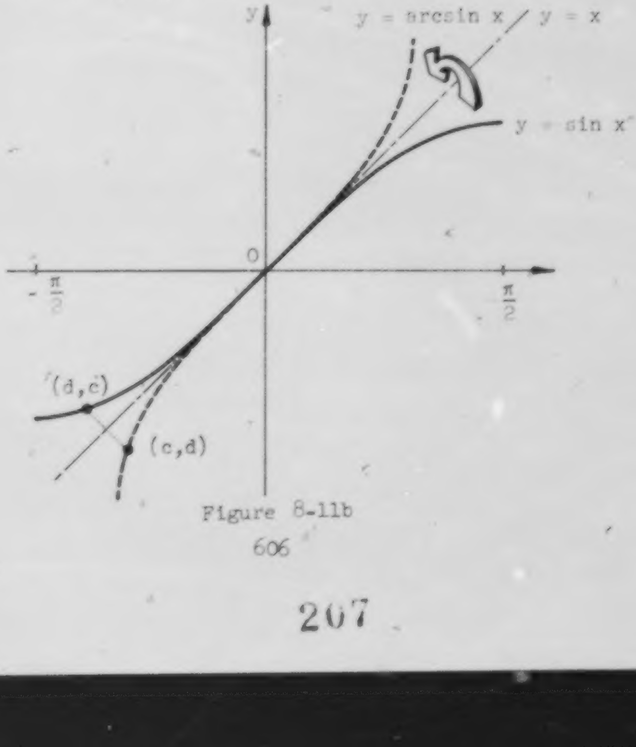


Figure 8-11b

Using the inverse function rule (5), we can express the derivative of the arcsine function f in terms of the sine function g. We have

$$f'(c) = \frac{1}{g'(f(c))}, \quad \text{if} \quad g'(f(c)) > 0.$$

In this case $g' : x \to \cos x$, so that

$$g'(f(c)) = \cos(\arcsin c)$$

and we have

$$f'(c) = \frac{1}{\cos(\arcsin c)}, \quad \text{if} \quad \cos(\arcsin c) > 0.$$

Referring to Figure 8-11b we see that

$$\cos(\arcsin c) = \sqrt{1 - c^2}$$

and hence we have

$$f'(c) = \frac{1}{\sqrt{1 - c^2}} \quad \text{if} \quad |c| < 1;$$

that is,

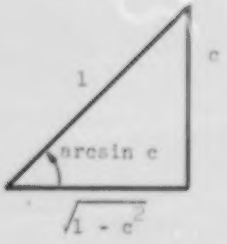


Figure 8-11b

$$(7) \qquad \boxed{D(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}, \quad \text{if} \quad |x| < 1.}$$

Taking the Chain Rule into account, we write the more general result:

$$\boxed{D(\arcsin u) = \frac{Du}{\sqrt{1 - u^2}}, \quad |u| < 1.}$$

The graph of the arcsin function has a vertical tangent at $x = \pm 1$. This seems reasonalbe if we recall the fact that the sine function has a horizontal tangent at $x = \pm \frac{\pi}{2}$.

The integration formula corresponding to (7) is

$$(8) \qquad \boxed{\int \frac{1}{\sqrt{1 - x^2}}\, dx = \arcsin x, \quad |x| < 1.}$$

Thus for $|a| < \frac{\pi}{2}$ and $|b| < \frac{\pi}{2}$ the Fundamental Theorem gives

$$\arcsin b - \arcsin a = \int_a^b \frac{1}{\sqrt{1 - x^2}}\,dx.$$

Replacing b by t, a by 0, and using the fact that $\arcsin 0 = 0$, we have

(9) $$\arcsin t = \int_0^t \frac{1}{\sqrt{1 - x^2}}\,dx; \quad |t| < 1.$$

The Arctangent Function

The function g defined by

$$g(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

is strictly increasing and continuous. Furthermore, the range of g is the entire real line; that is, if c is any real number, then there is a number d, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that $g(d) = c$. The inverse function f, known as the arctangent function, is accordingly defined for all real numbers c as follows:

(10) $f(c) = \arctan c$ is the real number d between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, such that $\tan d = c$.

Graph of $y = \arctan x$ and $y = \tan x$ are sketched in Figure 8-11c.

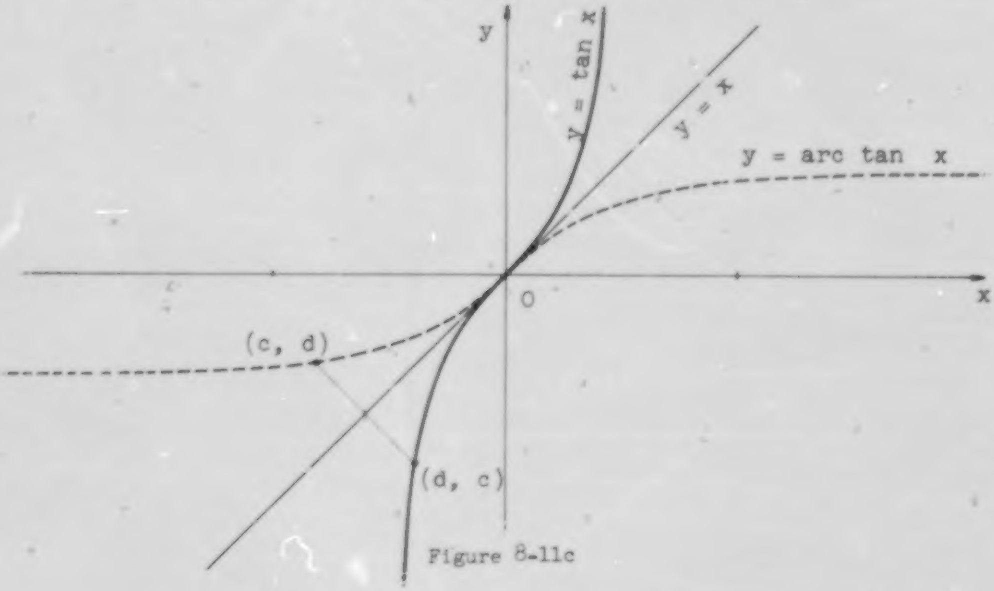


Figure 8-11c

The inverse function formula (5) gives

$$f'(c) = \frac{1}{g'(f(c))} = \frac{1}{\sec^2(\arctan c)}$$

since $D \tan x = \sec^2 x$. Referring to Figure 8-11d, we see that

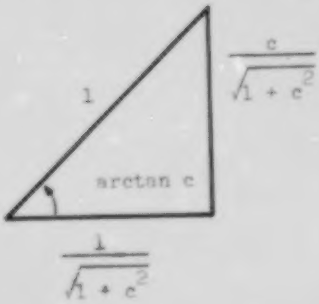


Figure 8-11d

$$\sec^2(\arctan c) = 1 + c^2$$

and hence

$$f'(c) = \frac{1}{1 + c^2} .$$

This fraction is always positive. In summary, we have

(11) $$\boxed{D(\arctan x) = \frac{1}{1 + x^2} ;}$$

and the corresponding integral form

(12) $$\boxed{\int \frac{1}{1 + x^2}\, dx = \arctan x.}$$

Taking the Chain Rule into account, we write the more general result:

$$\boxed{D \arctan u = \frac{Du}{1 + u^2}}$$

Exercises 8-11

1. Determine the domain and range and draw the graph of the function

   (a) $f : x \rightarrow \arcsin (\sin x)$

   (b) $f : x \rightarrow \sin (\arcsin x)$

   (c) $f : x \rightarrow \arcsin (\cos x)$

   (d) $f : x \rightarrow \cos (\arcsin x)$

   (e) $f : x \rightarrow \arctan (\tan x)$

2. Derive the formula

$$D \arccos x = \frac{-1}{\sqrt{1 - x^2}}$$

3. Derive each of the following formulas.

   (a) $D \operatorname{arccot} x = - \frac{1}{1 + x^2}$

   (b) $D \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}}$

   (c) $D \operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2 - 1}}$

4. Evaluate:

   (a) $D (\arcsin x + \arccos x)$

   (b) $D (x^2 \arcsin x)$

   (c) $D \frac{x^2}{\arctan x}$

   (d) $D (\arcsin x)^3$

   (e) $D (\frac{1}{1 + \arcsin x})$

   (f) $D (\frac{1 - \arctan x}{1 + \arctan x})$

5. Find $\lim_{h \rightarrow 0} \frac{\arcsin h}{h}$. (Hint: What is the definition of the derivative of $f(x) = \arcsin x$ at $x = 0$?)

6. Find $\frac{dy}{dx}$ if

   (a) $y = \arcsin x^2$

   (b) $y = \arctan (3x + 2)$

   (c) $y = e^{\arcsin x}$

   (d) $y = e^{2x} \arcsin (\frac{1}{x})$

7. Evaluate

(a) $\int_0^1 \frac{1}{1 + x^2}\,dx$

(b) $\int_{-\pi/4}^{\pi/6} \frac{1}{\sqrt{1 - t^2}}\,dt$

8. Find $F'(x)$ if $F(x)$ is given by

(a) $\int_0^x \frac{2}{1 + t^2}\,dt$

(b) $\int_0^{x^3} \frac{3}{\sqrt{1 - t^2}}\,dt$

(c) $\int_0^{\sin x} \frac{1}{1 + t^2}\,dt$

9. What is $\lim_{n \to \infty} \int_0^n \frac{1}{1 + t^2}\,dt$?

10. Show that each of the following functions g has an inverse f and find the derivative of f.

(a) $g : x \to \frac{1 - x}{1 + x}$, $x > -1$

(b) $g : x \to x|x|$ (a sketch is helpful.)

11. Show that if f is the inverse of g then $f(g(x)) = x$ for all x in the domain of g. Assuming that f and g are differentiable apply the chain rule to obtain a formula for the derivative of f. Is this the same as the rule (5)?

12. Suppose f_1 and f_2 are the respective inverses of g_1 and g_2. Let g be the function defined by $g(x) = g_1(g_2(x))$.

(a) Find an expression for the inverse of g.

(b) Use this method to find the inverse f of $x \to (3x + 2)^2$, $x \geq -\frac{2}{3}$.

(c) What is the derivative of the function f of part (b)?

13. Suppose f is the inverse of g. Put $y = g(x)$, $x = f(y)$. Show that

$$\left.\frac{dx}{dy}\right|_{y=s} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=f(s)}}$$

(The symbol $\left.\frac{ds}{dt}\right|_{t=\alpha}$ means the value of the derivative of s, considered as a function of t at the point where $t = \alpha$). This is the basis for the mnemonic expression of the inverse rule: $\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$.

14. The notation of Number 13 gives a method for finding derivatives. For example if $y = \arcsin x$, then $x = \sin y$ so $\frac{dx}{dy} = \cos y$ and hence

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}}$$

Use this method to find the derivative of

(a) $y = \arctan x$

(b) $y = \log_e x$

(c) $y = \sqrt{x}$

(d) $y = x^\pi$

8-12. Implicitly Defined Functions

A function which is described in terms of rational operations on, and compositions and inverses of, known functions is said to be defined explicitly.

For example, if

(1) $$y = f(x) = \sqrt{25 - x^2}, \quad |x| \leq 5,$$

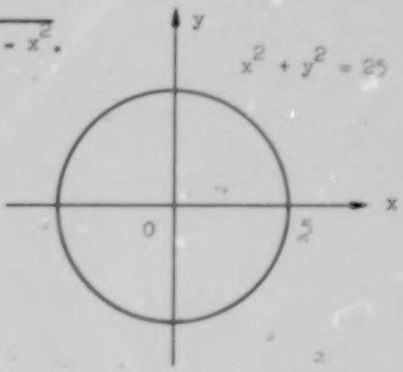
f is defined explicitly.

It often happens that a function is defined indirectly or implicitly. Thus

(2) $$x^2 + y^2 = 25$$

with the restriction that $y \geq 0$, defines the same function as

$$f : x \to \sqrt{25 - x^2}.$$



If we add no restriction, the graph of (2) is the circle with radius 5 and center at (0,0). Only the upper half of this circle is the graph of $y = \sqrt{25 - x^2}$. (The lower half is the graph of $y = -\sqrt{25 - x^2}$).

We can, of course, find $f'(x)$ from (1). In fact,

(3) $$f'(x) = \frac{-2x}{2\sqrt{25 - x^2}} = \frac{-x}{\sqrt{25 - x^2}}, \quad |x| < 5.$$

However, we can also find the slope of the graph from (2) without solving for y. First of all, $Dx^2 = 2x$. When we come to y^2 we note that this is really $[f(x)]^2$ so that by the Chain Rule, its derivative at x is

$$2f(x)\, f'(x).$$

Hence, we have

$$2x + 2f(x)\, f'(x) = D(25) = 0$$

and, therefore,

$$f'(x) = \frac{-2x}{2f(x)} = \frac{-x}{f(x)} \; (f(x) \neq 0).$$

Usually we simplify the notation and write

$$2x + 2yy' = 0$$

and

$$y' = -\frac{x}{y}, \quad y \neq 0, \tag{4}$$

leaving the result in terms of x and y. Of course, since $y = \sqrt{25 - x^2}$ on the upper semicircle, (4) is equivalent to (3). Often, however, we leave the answer in the form (3). If we wish the slope at $(3,4)$, say, (a point which is surely on the upper semicircle), we obtain

$$y' = -\frac{3}{4}.$$

Note that the tangent is perpendicular to the radius whose slope is $\frac{4}{3}$, which agrees with our geometrical knowledge.

There are many cases in which it would be difficult if not impossible to solve explicitly for y in terms of x.

Example 8-12a. Given

$$x^3 + y^3 = xy$$

with the point $(1,1)$ on its graph. We can find the slope y' there without difficulty but would find it very hard to do explicitly. We have

$$3x^2 + 3y^2y' = xy' + y.$$

Hence,

$$(3y^2 - x)y' = y - 3x^2$$

and

$$y' = \frac{y - 3x^2}{3y^2 - x},$$

so long as

$$3y^2 - x \neq 0.$$

At $(1,1)$,

$$y' = \frac{1 - 3}{3 - 1} = -1.$$

Example 8-12b. Given $x^3y + xy^2 = 6$ to find y' at the point $(1,2)$. We find

$$x^3y' + 3x^2y + x\,2yy' + y^2 = 0.$$

Then

$$(x^3 + 2xy)y' = -(3x^2y + y^2),$$

$$y' = \frac{-(3x^2y + y^2)}{x^3 + 2xy}.$$

At $(1,2)$,

$$y' = \frac{-10}{5} = -2.$$

It is possible to solve for y by the quadratic formula. Thus

$$y = \frac{-x^3 \pm \sqrt{x^6 + 24x}}{2x}.$$

Which sign must we choose so that $y = 2$ when $x = 1$? We forbear to find y', since from here on the direct method becomes too painful.

Implicit differentiation often simplies the calculations involved in problems about related rates (Section 8-8).

Example 8-12c. Recall Example 8-8a. Let s be the length of the line-of-sight to the horizon, and h the height of the helicopter. Then

$$s^2 + 4000^2 = (h + 4000)^2.$$

Differentiating implicitly with respect to t, we obtain

$$(1) \qquad 2s\frac{ds}{dt} = 2(h + 4000)\frac{dh}{dt}.$$

When $t = \frac{1}{10}$, $h = 2$, and

$$s^2 = 4002^2 - 4000^2 = 16002$$

so

$$s = \sqrt{16002} \approx 126.5.$$

$\frac{dh}{dt} = 20$, the upward rate of the helicopter. Substituting in (1), we obtain

$$2(126.5)\left.\frac{ds}{dt}\right|_{t=\frac{1}{10}} = 2(4002)\cdot 20,$$

and
$$\left.\frac{ds}{dt}\right|_{t=\frac{1}{10}} = \frac{4002 \cdot 20}{126.5} \approx 633.$$

Example 8-12d. Recall Example 8-8h. Let h represent the distance the Columbia falls in t hours, and let x represent the distance traveled by the carrier in the same amount of time. Then

$$s^2 = h^2 + x^2.$$

Differentiating implicitly with respect to t, we have

$$2s\,\frac{ds}{dt} = 2h\,\frac{dh}{dt} + 2x\,\frac{dx}{dt}$$

or
$$s \cdot \frac{ds}{dt} = h \cdot \frac{dh}{dt} + x\,\frac{dx}{dt}\,.$$

$\frac{dh}{dt} = -20$ and $\frac{dx}{dt} = -30$, the velocities of Columbia and the carrier, respectively. Negative signs are included to indicate that h and x are decreasing as t increases. One minute before splashdown, $t = \frac{1}{30}$ and

$$h = \frac{1}{3} \text{ (the distance the Columbia falls in one minute)}$$

$$x = 9 - 30 \cdot \frac{1}{30} = 8$$

and
$$s = \sqrt{h^2 + x^2} = \sqrt{64\tfrac{1}{9}}$$

$$\therefore \left.\frac{ds}{dt}\right|_{t=\frac{1}{30}} = \frac{h \cdot \frac{dh}{dt} + x\,\frac{dx}{dt}}{s} = \frac{\frac{1}{3}(-20) + 8(-30)}{\sqrt{64\tfrac{1}{9}}} \approx -30.8.$$

Hence, the distance between the Columbia and the carrier was decreasing at the rate of 30.8 mi./hr. at 9:49 a.m.

Exercises 8-12

1. For positive x, if $y = x^r$, where r is a rational number, say $r = \frac{p}{q}$ (p, q integers), then $y^q = x^p$. Assuming the existence of the derivative Dy, derive the formula $Dy = rx^{r-1}$ using implicit differentiation and the differentiation formula $Dx^n = nx^{n-1}$, for integral n.

2. For each of the following, find y' without solving for y as a function of x.

   (a) $5x^2 + y^2 = 12$

   (b) $2x^2 - y^2 + x - 4 = 0$

   (c) $y^2 - 3x^2 + 6y = 12$

   (d) $x^3 + y^3 - 2xy = 0$

3. For each of the following use implicit differentiation to find Dy.

   (a) $x^2 = \dfrac{y - x}{y + x}$

   (b) $x^2y + xy^2 = x^3$

   (c) $x^my^n = 10$ (m, n integers)

   (d) $\sqrt{xy} + x = y^{-1}$

4. For each equation, find the slope of the curve represented, at the stated point.

   (a) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$ at the point $(-2,1)$

   (b) $x^3 + y^2x^2 + y^3 - 1 = 0$ at the point $(1,-1)$

   (c) $x^2 - x\sqrt{xy} - 6y^2 = 2$ at the point $(4,1)$

   (d) $x \cos y = 3x^2 - 5$ at the point $(\sqrt{2}, \frac{\pi}{4})$

5. For each equation, find the slope of the curve represented at the point or points where $x = y$. Give a geometric explanation for these results.

   (a) $x^3 - 3axy + y^3 = 0$

   (b) $x^m + y^m = 2$

   (c) $x^2 + y^2 = 2axy$

6. Find y' by implicit differentiation.

   (a) $a \sin y + b \cos x = 0$

   (b) $x \cos y + y \sin x = 0$

   (c) $\sin xy = \sin x + \sin y$

   (d) $\csc(x + y) = y$

   (e) $x \tan y - y \tan x = 1$

   (f) $y \sin x = x \tan y$

   (g) $xy + \sin y = 5$

7. If $0 < x < a$, then the equation $x^{1/2} + y^{1/2} = a^{1/2}$ defines y as a function of x. Assuming the existence of the derivative, show without solving for y that $f'(x)$ is always negative.

8. A spherical balloon is being filled with helium at the rate of 100 cubic inches/min. How fast is the radius increasing when it has reached the value of 5 inches? [Use implicit differentiation.]

9. Water is leaking out of a conical tank at the rate of 3 ft.^{3}/min. The tank is 30 ft. across at the top and 10 ft. deep. How fast is the water level dropping when the depth reaches 4 feet? [The volume of a cone is $\frac{1}{3}$(altitude) $\cdot$ (area of base).]

10. A trough 10 feet long has a cross section the same shape as an isosceles trapezoid with altitude 2 ft., upper base 3 ft. and lower base 1 ft. If water is poured in at the rate of 5 ft.^{3}/min., how fast is the water level rising when the water is 1 ft. deep?

11. (a) Find $\frac{dy}{dx}$ if $x^2 + y^2 = 2xy + 1$.

    (b) Sketch the graph of $x^2 + y^2 = 2xy + 1$.

    (c) Sketch the graph of $|x - y| = 1$.

12. Work Exercises 8-8, Number 17 using implicit differentiation.

Chapter 9

INTEGRATION THEORY AND TECHNIQUE

We now return to our study of integration, begun in Chapter 7. We saw that the area bounded by the graph of a function f, the x-axis and vertical lines at a and b, was given by $F(b) - F(a)$, where F is an antiderivative of f (the Fundamental Theorem of Calculus, Section 7-3). Various elementary antidifferentiation formulas and the use of tables were discussed in the final section of Chapter 7. In the first section of this chapter we present a method for extending the scope of these formulas and tables. This method is known as the method of substitution and is, in fact, the antidifferentiation form of the chain rule. By appropriate substitution many unfamiliar integrals can be converted into forms previously encountered or listed in the tables. More about the method of substitution and other methods of integration is contained in Appendix 4.

The Fundamental Theorem enables us to calculate areas (when antiderivatives can be found). There are other interpretations of the difference $F(b) - F(a)$ where $F' = f$. One of these interpretations is discussed in this chapter. We show how the concept of average value of a function is related to integration (Section 9-2). Then we show how the average value interpretation can be used to calculate volumes of solids of revolution (Section 9-3).

Numerical methods for approximating integrals are discussed in Section 9-4. These methods are useful, particularly in conjunction with high speed computers, in estimating integrals when antiderivatives cannot be found. The final section of this chapter shows how we can obtain Taylor approximations with error estimates by integrating inequalities.

9-1. The Method of Substitution

The scope of our integration tables can be greatly extended by using the method of substitution. This method often enables us to transform unfamiliar integrals into familiar ones. It is based upon an integral form of the chain rule.

In terms of antiderivatives, we have learned to symbolize the derivative statement

$$\frac{d\,F(u)}{du} = f(u)$$

by writing

$$F(u) = \int f(u)du. \tag{1}$$

If u is a function of x, the chain rule shows that

$$\frac{d\,F(u)}{dx} = \frac{d\,F(u)}{du} \cdot \frac{du}{dx} = f(u) \cdot \frac{du}{dx}\,, \tag{2}$$

which similarly justifies the statement that

$$F(u) = \int f(u)\,\frac{du}{dx}\,dx. \tag{3}$$

Together, (2) and (3) show that

$$\boxed{\int f(u)du = \int f(u)\frac{du}{dx}\,dx,} \tag{4}$$

if u is a function of x.

This equality (4) vastly increases the number of antiderivatives we may determine. It often happens that we are confronted by a rather complicated integral in terms of x, say, which becomes substantially simplified and familiar if we can express it in terms of a suitable variable u which is a function of x.

For example, suppose we seek to determine the antiderivative

$$\int 2x \cos x^2\,dx.$$

If we let

$$u = x^2,$$

and

$$\frac{du}{dx} = 2x,$$

then

$$\int 2x \cos x^2\,dx = \int \cos u\,\frac{du}{dx}\,dx.$$

According to (4), with $f(u) = \cos u$, we may conclude that

$$\int \cos u\,\frac{du}{dx}\,dx = \int \cos u\,du,$$

and we should recognize this antiderivative as $\sin u$. Hence,

$$\int 2x \cos x^2 \, dx = \int \cos u \, du = \sin u$$

and upon substituting back $u = x^2$, we find

$$\int 2x \cos x^2 \, dx = \sin x^2,$$

as desired.

The Leibniz notation, $\frac{dy}{dx}$, is more than a convenient device for remembering the chain rule and the substitution rule (4). It prompts mathematicians in practice to deal with the "numerator," dy, and the "denominator," dx, as if $\frac{dy}{dx}$ were a common fraction. For example, equation (4) suggests that operationally the symbol

$$\frac{du}{dx} dx$$

may be replaced by the symbol

$$du$$

when we perform substitutions to integrate a function f. The symbols "dx," "du," "dy," etc., are called differentials. In practice they short cut the thinking required to evaluate integrals by the method of substitution, as the examples below indicate.

To find suitable substitutions to reduce an integral to a known form is no easy task and, in fact, may not be possible (see Example 9-1a, below). Practice is required to obtain skill at making appropriate substitutions.

Example 9-1a. Find $\int xe^{x^2} dx$.

Put $u = x^2$, so that $\frac{du}{dx} = 2x$ and hence,

$$\frac{1}{2} du = x \, dx.$$

Upon writing

$$\int xe^{x^2} dx = \int e^{x^2} (x \, dx)$$

we can make the replacements

$$(5) \qquad u = x^2 \quad \text{and} \quad \frac{1}{2}\,du = x\,dx$$

to obtain

$$\begin{aligned}\int x\,e^{x^2}\,dx = \int e^{x^2}\,(x\,dx) &= \int e^u(\tfrac{1}{2}\,du)\\ &= \frac{1}{2}\int e^u\,du\\ &= \frac{1}{2}\,e^u.\end{aligned}$$

Now replace u by x^2 to obtain

$$\int xe^{x^2}\,dx = \frac{1}{2}\,e^{x^2}.$$

The formal substitutions (5) and the equation

$$\int x\,e^{x^2}\,dx = \frac{1}{2}\int e^u du$$

are a shorthand for the statements

$$\int x\,e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\,2x\,dx = \frac{1}{2}\int e^u\,\frac{du}{dx}\,dx$$

where $u = x^2$.

Example 9-1b. Find $\int \sin\,(2x + 3)dx$.

Put

$$u = 2x + 3,\ \frac{du}{dx} = 2.$$

Make the substitutions

$$u = 2x + 3,\ \frac{1}{2}\,du = dx$$

to obtain

$$\begin{aligned}\int \sin(2x+3)dx &= \int (\sin u)\tfrac{1}{2}\,du \\ &= \tfrac{1}{2}\int \sin u\,du \\ &= -\tfrac{1}{2}\cos u \\ &= -\tfrac{1}{2}\cos(2x+3).\end{aligned}$$

In general, we have seen that replacing x by $ax + b$ multiplies the derivative by a. Thus replacement of x by $ax + b$ multiplies the antiderivative by $\frac{1}{a}$, that is

$$(6) \qquad \boxed{\text{If } F(x) = \int f(x)dx, \text{ then } \frac{1}{a}F(ax+b) = \int f(ax+b)dx}$$

Example 9-1c. Find $\int \tan x\,dx$.

Since

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad D\cos x = -\sin x$$

it seems appropriate to try the substitution

$$u = \cos x.$$

Then $\frac{du}{dx} = -\sin x$, so that $-du = \sin x\,dx$ and we have

$$\begin{aligned}\int \tan x\,dx &= \int \frac{1}{\cos x}(\sin x\,dx) \\ &= \int \frac{1}{u}(-du) \\ &= -\int \frac{1}{u}\,du \\ &= -\log_e u, \quad \text{if } u > 0 \\ &= -\log_e(\cos x), \quad \text{if } \cos x > 0.\end{aligned}$$

The result

$$\int \tan x\,dx = -\log_e(\cos x)$$

is formula 12 in the Table of Integrals. (See Exercises 9-1, No. 9, for a justification of the absolute value sign.)

Example 9-1d. Find $\int \frac{1}{x \log_e x}\,dx$.

Put $u = \log_e x$, so that

$$\frac{du}{dx} = \frac{1}{x}, \quad \text{that is,} \quad du = \frac{1}{x}\,dx.$$

Thus

$$\begin{aligned}\int \frac{1}{x \log_e x}\,dx &= \int \frac{1}{\log_e x}\left(\frac{1}{x}\,dx\right)\\ &= \int \frac{1}{u}\,du\\ &= \log_e u, \quad \text{if} \quad u > 0\\ &= \log_e(\log_e x), \quad \text{if} \quad \log_e x > 0.\end{aligned}$$

Example 9-1e. Find $\int \sin^2 x \cos x\,dx$.

We try

$$u = \sin x, \quad \text{so that} \quad \frac{du}{dx} = \cos x.$$

Making the substitutions

$$u = \sin x, \quad du = \cos x\,dx$$

thus gives

$$\begin{aligned}\int \sin^2 x \cos x\,dx &= \int u^2\,du\\ &= \frac{u^3}{3}\\ &= \frac{\sin^3 x}{3}.\end{aligned}$$

Example 9-1f. Find $\int_0^1 (1 - x^2)^5\, x\,dx$.

One way to do this is to carry out the indicated multiplications and calculate

$$\int_0^1 (x - 5x^3 + 10x^5 - 10x^7 + 5x^9 - x^{11})dx.$$

Let us, instead, try the substitution

$$u = 1 - x^2; \ \frac{du}{dx} = -2x, \quad \text{so that} \quad x\,dx = -\frac{1}{2}\,du$$

and we have

$$\int (1 - x^2)^5\, x\, dx = \int u^5(-\tfrac{1}{2}\,du) = -\frac{1}{2}\int u^5\, du$$

$$= \frac{-u^6}{12} = -\frac{(1 - x^2)^6}{12}.$$

Hence,

$$\int_0^1 (1 - x^2)x\, dx = -\frac{(1 - x^2)^6}{12}\Bigg|_0^1 = -\frac{(1 - 1)^6}{12} + \frac{(1 - 0)^6}{12} = \frac{1}{12}.$$

Note that replacing x by 0 and 1 in the expression

$$u = 1 - x^2$$

gives the respective values 1 and 0 for u and that

$$-\frac{1}{2}\int_1^0 u^5\, du = -\frac{u^6}{12}\Bigg|_1^0 = \frac{-0}{12} + \frac{1}{12} = \frac{1}{12}.$$

In other words, we can express the limits of integration in terms of u and complete our calculation in terms of u. The next example also illustrates this fact.

Example 9-1g. Find the area bounded by

$$x = 0,\ x = 1,\ y = 0 \quad \text{and} \quad y = x\sqrt{x^2 + 1}.$$

In integral notation our problem is to find

$$\int_0^1 x\,\sqrt{x^2 + 1}\, dx.$$

Put

$$u = x^2 + 1, \ \frac{du}{dx} = 2x, \ \frac{1}{2}\,du = x\, dx$$

so that

$$u = 1 \quad \text{when} \quad x = 0 \quad \text{and} \quad u = 2 \quad \text{when} \quad x = 1.$$

Substitution of these gives

$$\int_0^1 x\sqrt{x^2+1}\,dx = \int_1^2 u^{1/2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_1^2 u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^2 = \frac{1}{3}u^{3/2}\Big|_1^2$$

$$= \frac{1}{3}[2^{3/2} - 1^{3/2}] = \frac{1}{3}(2\sqrt{2} - 1)$$

Example 9-1h. Find $\int_{10}^{1} t^3 e^{-t^2}\,dt$.

Let

$$u = -t^2, \quad \frac{du}{dt} = -2t, \quad -\frac{1}{2}\,du = t\,dt$$

so that $u = -1$ when $t = 1$ and $u = -100$ when $t = 10$

$$\int_{10}^{1} t^3 e^{-t^2}\,dt = \int_{10}^{1} t^2 e^{-t^2}(t\,dt) = \int_{-100}^{-1} (-u)e^u\left(-\frac{1}{2}\,du\right)$$

$$= \frac{1}{2}\int_{-100}^{-1} ue^u\,du.$$

The Tables give

$$\int ue^u du = ue^u - e^u$$

so that

$$\int_{10}^{1} t^3e^{-t^2}\,dt = \frac{1}{2}(ue^u - e^u)\Big|_{-100}^{-1}$$

$$= \frac{1}{2}[((-1)e^{-1} - e^{-1}) - ((-100)e^{-100} - e^{-100})]$$

$$= \frac{101}{2}e^{-100} - e^{-1}$$

Example 9-1i. Find $\int \frac{1 + 3x^2}{x + x^3}\,dx$.

If we put

$$u = x + x^3, \quad \text{then} \quad \frac{du}{dx} = 1 + 3x^2$$

so that

$$\begin{aligned}\int \frac{1 + 3x^2}{x + x^3}\,dx &= \int \frac{1}{x + x^3}((1 + 3x^2)dx) \\ &= \int \frac{1}{u}\,du \\ &= \log_e u, \quad \text{if} \quad u > 0, \\ &= \log_e(x + x^3), \quad \text{if} \quad x + x^3 > 0.\end{aligned}$$

Example 9-1j. Find $\int_0^a \frac{x^2}{\sqrt{1 - x^6}}\,dx, \quad |a| < 1$

Let us try the substitution

$$u = 1 - x^6, \quad \frac{du}{dx} = -6x^5.$$

We can then write

$$\begin{aligned} x &= (1 - u)^{1/6}, \quad \text{so that} \quad x^5 = (1 - u)^{5/6} \\ x^2 &= (1 - u)^{1/3}, \quad \text{and} \quad -\tfrac{1}{6}(1 - u)^{-5/6}\,du = dx.\end{aligned}$$

Hence

$$\int \frac{x^2}{\sqrt{1 - x^6}}\,dx = \int \frac{(1 - u)^{1/3}}{u^{1/2}}\left(-\tfrac{1}{6}(1 - u)^{-5/6}\right)du.$$

This latter integral appears to be quite complicated, so let us try another substitution.

Put

$$u = x^3, \quad \frac{du}{dx} = 3x^2\,dx, \quad \frac{1}{3}\,du = x^2\,dx.$$

This gives

$$\int_0^a \frac{x^2}{\sqrt{1 - x^6}}\,dx = \int_0^{a^3} \frac{1}{\sqrt{1 - u^2}}\left(\frac{1}{3}\,du\right)$$

$$= \frac{1}{3}\arcsin u \,\Big|_0^{a^3}$$

$$= \frac{1}{3}\arcsin a^3$$

Example 9-1k. Find $\int_0^1 \frac{x^2}{\sqrt{1 + x}}\,dx$.

Try the substitution

$$u = 1 + x, \quad du = dx$$

to obtain

$$\int_0^1 \frac{x^2}{\sqrt{1 + x}}\,dx = \int_1^2 \frac{(u - 1)^2}{\sqrt{u}}\,du$$

$$= \int_1^2 \frac{u^2 - 2u + 1}{\sqrt{u}}\,du = \int_1^2 (u^{3/2} - 2u^{1/2} + u^{-1/2})du$$

$$= \left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2}\right)\Big|_1^2$$

$$= \frac{14}{15}\sqrt{2} - \frac{16}{15}\,.$$

Other substitutions are also useful. For example, put

$$u = \sqrt{1 + x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{1 + x}}$$

so that $2u\,du = dx$, $x^2 = (u^2 - 1)^2$ and

$$\int_0^1 \frac{x^2}{\sqrt{1 + x}}\,dx = \int_1^{\sqrt{2}} \frac{(u^2 - 1)^2}{u}(2u\,du)$$

$$= 2\int_1^{\sqrt{2}} (u^2 - 1)^2\,du = 2\int_1^{\sqrt{2}} (u^4 - 2u^2 + 1)du$$

$$= 2\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right)\Big|_1^{\sqrt{2}} = \frac{14}{15}\sqrt{2} - \frac{16}{15}$$

Example 9-14. Find $\int e^{-x^2}\,dx$.

We first try

$$u = x^2, \quad du = 2x\,dx$$

obtaining

$$\int e^{-x^2}\,dx = \int \frac{e^{-u}}{2\sqrt{u}}\,du.$$

The latter seems to be more complicated. Searching tables of integrals leads us nowhere for we find neither expression in our tables. We could try other substitutions, such as $u = \sqrt{x}$, or even make wilder stabs, such as $u = \sin x$.

In fact, no matter what substitution we trv we shall get nowhere, for it was shown by Liouville in 1835 that the integral of e^{-x^2} cannot be expressed as a finite combination or composition of polynomials, circular functions, exponential functions, or logarithms.

Exercises 9-1

1. Use the indicated substitution to find each of the following: (Wherever they appear, a, b, and c represent non-zero constants.)

   (a) $\int \frac{x^2}{x^3 + a^3}\,dx; \quad u = x^3 + a^3$

   (b) $\int x^3 \sqrt[7]{1 - x^4}\,dx; \quad u = 1 - x^4$

   (c) $\int \frac{(a + b\sqrt{x})^{13}}{\sqrt{x}}\,dx, \; b \neq 0; \quad u = a + b\sqrt{x}$

   (d) $\int \frac{x^2 + 1}{x - 1}\,dx; \quad u = x - 1$

   (e) $\int \frac{x}{x^2 + a^2}\,dx; \quad u = x^2 + a^2$

   (f) $\int \frac{x}{x^4 + a^2}\,dx, \; a \neq 0; \quad u = \frac{x^2}{a}$

   (g) $\int (\cos x)e^{\sin x}\,dx; \quad u = \sin x$

   (h) $\int \frac{ae^x}{b + ce^x}\,dx, \; c \neq 0; \quad u = b + ce^x$

   (i) $\int \sec x\,dx; \quad u = \sec x + \tan x$

2. Find each of the following integrals by making an appropriate linear substitution.

   (a) $\int e^{2x}\,dx$

   (b) $\int (1 - \frac{1}{2}x)^{10}\,dx$

   (c) $\int \sin ax\,dx$

   (d) $\int \sqrt[4]{3x + 1}\,dx$

   (e) $\int \frac{1}{2 - 3x}\,dx$

   (f) $\int \frac{1}{\sqrt{(1 - 5x)^3}}\,dx$

   (g) $\int \frac{1}{a^2 + x^2}\,dx$

   (h) $\int \tan(\frac{1}{2}x - 3)dx$

3. Find each of the following integrals by making an appropriate substitution.

(a) $\int (4 - 3x^2)^6 \, x \, dx$

(b) $\int \cos^5 x \sin x \, dx$

(c) $\int \sin^2 2x \cos 2x \, dx$

(d) $\int \frac{e^{1/x}}{x^2} \, dx$

(e) $\int x \sqrt{1 + 4x^2} \, dx$

(f) $\int \frac{(\log_e x)^2}{x} \, dx$

(g) $\int \frac{\cos \sqrt{2x}}{\sqrt{x}} \, dx$

(h) $\int \frac{\sin x}{(a + b \cos x)^2} \, dx$

(i) $\int \frac{x^2}{(4x^3 - 1)^{3/2}} \, dx$

(j) $\int \frac{x}{1 + x^2} \, dx$

(k) $\int \frac{x}{1 + x^4} \, dx$

(ℓ) $\int \frac{x}{\sqrt{1 - 9x^4}} \, dx$

(m) $\int \sin^2 x \cos^3 x \, dx$ [Hint: Write $\cos^3 x = \cos^2 x \cos x$
$= (1 - \sin^2 x)\cos x$.]

(n) $\int \sin^3 4x \cos^8 4x \, dx$ [Hint: Rewrite $\sin^3 4x$.]

4. Evaluate each of the following

(a) $\int_2^3 \frac{1}{(2x + 1)^2} \, dx$

(b) $\int_0^{\pi} \cos^4 x \sin x \, dx$

(c) $\int_0^{\pi/3} \cos 4x \, dx$

(d) $\int_{-1/2}^{0} (2x + 1)^{17} \, dx$

(e) $\int_1^0 \sqrt{1 + x} \, dx$

(f) $\int_{-1}^{1} x \sqrt{1 - x^2} \, dx$

(g) $\int_0^{1/2} \frac{1}{1 + 4x^2} \, dx$

(h) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{9 - x^2}} \, dx$

(i) $\int_1^2 \frac{\log_e x}{x} \, dx$

(j) $\int_0^1 x^3 \sqrt{1 - x^2} \, dx$

(k) $\int_{-1}^{1} x^2 e^{x^3} \, dx$

(ℓ) $\int_0^{\sqrt{\frac{\pi}{2}}} x \sin (2x^2) dx$

5. Sometimes it is useful to reduce an integral to a known integral by making two or more separate substitutions. For example to find

$$\int \frac{2e^x}{4 + e^{2x}}\,dx$$

we might put $u = e^x$, $du = e^x\,dx$ to obtain

$$\int \frac{2e^x}{4 + e^{2x}}\,dx = \int \frac{2}{4 + u^2}\,du$$

and then put $v = \frac{u}{2}$, $2dv = du$ to obtain

$$\int \frac{2}{4 + u^2}\,du = \int \frac{4}{4 + 4v^2}\,dv = \int \frac{1}{1 + v^2}\,dv$$

$$= \arctan v = \arctan \frac{u}{2}$$

$$= \arctan \left(\frac{e^x}{2}\right).$$

Find:

(a) $\displaystyle\int \frac{\cos x}{\sqrt{9 - \sin^2 x}}\,dx$

(b) $\displaystyle\int \frac{x^2}{2 + x^6}\,dx$

(c) $\displaystyle\int \frac{1}{\sqrt{x} + x}\,dx$

6. Find each of the following by making appropriate substitutions and then using a table of integrals.

(a) $\displaystyle\int x^2 \sin (x - 1)dx$

(b) $\displaystyle\int_0^2 x\, e^{2x}\,dx$

(c) $\displaystyle\int_0^{\pi} x \sin 3x\,dx$

(d) $\displaystyle\int x \cos^3 (x^2)dx$

(e) $\displaystyle\int x^3\, e^{-4x}\,dx$

(f) $\displaystyle\int x\, e^{x^2} \sin 2x^2\,dx$

(g) $\displaystyle\int_0^1 x^2 \log_e (x + 1)dx$

(h) $\displaystyle\int (\sin x) \log_e (\cos x)dx$

(i) $\int \sin(x + 1)\cos(2x + 2)dx$ (k) $\int \frac{e^x}{4e^{2x} - 2e^x + 1} dx$

(j) $\int \frac{\sin x}{2\cos^2 x + \cos x - 3} dx$ (ℓ) $\int \frac{1}{x\sqrt{(\log_e x)^2 + 1}} dx$

7. Even though $\int e^{-x^2} dx$ cannot be expressed in terms of elementary functions, approximate values of the definite integral $\int_a^b e^{-x^2} dx$ can be found (using, for example, the methods of Section 9-4). Related integrals can then be evaluated by appropriate substitutions. Suppose

$$\int_0^1 e^{-x^2} dx = \alpha.$$

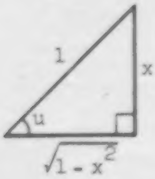
Show that

(a) $\int_{-1}^{0} e^{-x^2} dx = \alpha$ (c) $\int_{-1}^{3} e^{-\frac{(x-1)^2}{4}} dx = 4\alpha$

(b) $\int_{-1}^{1} e^{-x^2} dx = 2\alpha$ (d) $\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx = 2\alpha$

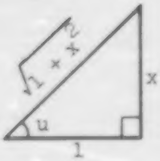
8. (a) Use the substitution $x = \sin u$ to find

$$\int_0^1 \sqrt{1 - x^2}\, dx.$$



(b) Find $\int \frac{\sqrt{1 + x^2}}{x^4} dx.$

Let $x = \tan u$.



A complete discussion of this method of substitution is given in Appendix 4.

9. (a) From Chapter 7 we know that if $x > 0$,

$$\int \frac{1}{x}\,dx = \log_e x.$$

Show that if $x < 0$, then

$$\int \frac{1}{x}\,dx = \log_e |x|$$

by substituting $t = -x$ in place of x.

(b) In the Table of Integrals the result of part (a) is given as formula 2:

$$\int \frac{1}{x}\,dx = \log_e |x|$$

This formula can be meaningfully applied to calculate $\int_a^b \frac{1}{x}\,dx$ only if a and b have the same sign. Why? (See Exercises 7-6, No. 37.)

9-2. The Average Ordinate, or Mean Value, of a Function

As we have seen, one possible interpretation of $\int_a^b f$ is the (signed) area A shown in Figure 9-2a.

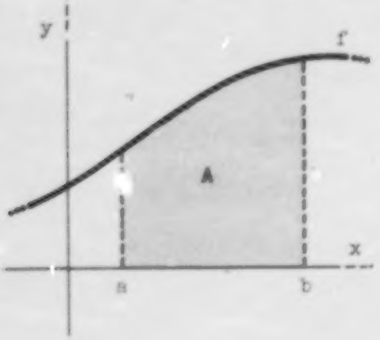


Figure 9-2a

The average value of $f(x)$ on the interval $[a,b]$ is thought of as the height of the rectangle with base $(b - a)$ whose area equals $\int_a^b f$. In Figure 9-2b, it is denoted $f(x)_{av.}$

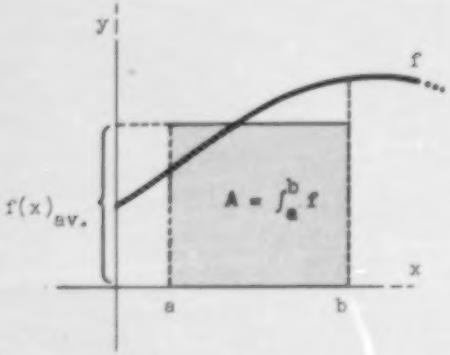


Figure 9-2b

Thus, we define $f(x)_{av.}$ by

(1) $$(b - a) \cdot f(x)_{av.} = \int_a^b f$$

If $F' = f$, then

$$\int_a^b f = F(b) - F(a),$$

and equation (1) becomes

$$f(x)_{av.} = \frac{F(b) - F(a)}{b - a}.$$

The ratio $\frac{F(b) - F(a)}{b - a}$ has been encountered previously as the average rate of change of the function F on the interval $a \le x \le b$.

For example, if $s = F(t)$ represents the distance of a body from a fixed point at time t, then

$$\frac{F(b) - F(a)}{b - a}$$

is the average velocity in the time interval $a \le t \le b$. The derivative $F' = f$ can then be interpreted as the velocity function of the motion; that is, $F'(t) = f(t)$ is the velocity of the body at time t. Thus the integral

$$\frac{1}{b - a}\int_a^b f = \frac{F(b) - F(a)}{b - a}$$

is the average velocity of the motion in the time interval $a \le t \le b$.

In general, no matter what the interpretation for a particular function f, the number

$$\frac{1}{b - a}\int_a^b f \tag{2}$$

is called the average value of f on the interval. This interpretation of (2) is very useful. In the next section we will see how the concept of average value is related to volumes of solids of revolution. Averaging ideas also lead us to useful methods for approximating integrals (see Section 9-4).

Example 9-2a. Suppose an automobile travels between two points 100 miles apart, traveling at an average speed of 30 miles per hour for the first half hour, then at an average of C miles per hour for the remainder of the trip. What must C be in order that the trip shall take two hours?

Let $f(t)$ denote the velocity of the automobile at time t. While we do not know f explicitly we do know its average value on the interval $0 \le t \le 2$ and on each of the subintervals $0 \le t \le \frac{1}{2}$, and $\frac{1}{2} \le t \le 2$. These are respectively:

$$\frac{1}{2-0}\int_0^2 f = \frac{100}{2} = 50$$

$$\frac{1}{\frac{1}{2}-0}\int_0^{1/2} f = 30$$

and

$$\frac{1}{2-\frac{1}{2}}\int_{1/2}^{2} f = C = \frac{1}{\frac{3}{2}}\left[\int_0^2 f - \int_0^{1/2} f\right]$$

Since $\int_0^2 f = 100$ and $\int_0^{1/2} f = 15$,

$$C = \frac{2}{3}(100 - 15) = \frac{170}{3} \approx 56.67$$

Hence, the speed we must average in the last $1\frac{1}{2}$ hours in order to average 50 mi./hr. for the trip is approximately 56.67 mi./hr.

Example 9-2b. Suppose $f : x \to \sin x$ and that g is the constant function $g : x \to c$. What must c be in order that the area bounded by the graph of f, $x = 0$, $x = \pi$ and $y = 0$ is the same as the area bounded by g, $x = 0$, $x = \pi$ and $y = 0$?

The situation is illustrated in Figure 9-2c. Our problem is to determine the height c of the shaded rectangle OABC so that its area is the same as the shaded area under the curve $y = \sin x$.

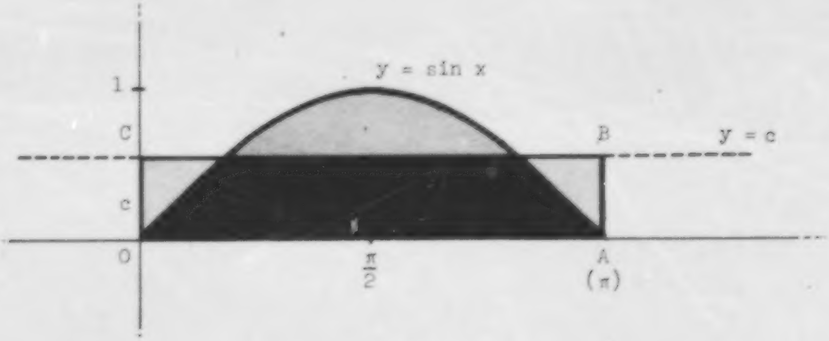


Figure 9-2c

The area of the shaded region under the curve $y = \sin x$ is

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -\cos \pi + \cos 0 = 2$$

while the area of the rectangle is

$$\int_0^{\pi} g = \int_0^{\pi} c \, dx = c(\pi - 0) = \pi c,$$

and therefore, $\pi c = 2$, that is

$$c = \frac{2}{\pi} .$$

The number $\frac{2}{\pi}$ is just the average value of f:

$$\frac{2}{\pi} = \frac{1}{\pi - 0} \int_0^{\pi} f .$$

Exercises 9-2

1. For each of the following sketch the graph of f and find its average value on the indicated interval.

   (a) $f : x \rightarrow 3x^2 + 4x - 7, \quad -1 \leq x \leq 0$

   (b) $f : x \rightarrow \frac{1}{1 + x^2}, \quad 0 \leq x \leq 1$

   (c) $f : s \rightarrow se^{s^2}, \quad -1 \leq s \leq 1$

   (d) $f : t \rightarrow \sqrt{2t + 1}, \quad -\frac{1}{2} \leq t \leq 4$

   (e) $f : x \rightarrow \frac{1}{3x + 1}, \quad 1 \leq x \leq 0$

2. Find the average value of the sine function on each of the following intervals.

   (a) $0 \leq x \leq \pi$

   (b) $1 + 7\pi \leq x \leq 1 + 9\pi$

   (c) $-\pi \leq x \leq \pi$

   (d) $c \leq x \leq c + 2\pi$, where c is any number.

3. Show that if f is periodic and integrable with period α, then the average value of f on any interval of length α is a constant, independent of the location of the interval. (See Exercise 2.)

4. Find the average value of the slope of the tangent to the graph of $x \rightarrow x^2 + 1$ in the interval $-1 \leq x \leq 3$.

5. Let f_{av} represent the average value of a function f on the interval $[0,1]$. For $f : x \rightarrow x^2$, show that

$$(f_{av})^2 \neq (f^2)_{av}.$$

6. Suppose a particle moves so that its acceleration at time t is $a(t) = t^3 + \frac{1}{\sqrt{t}}$. What is its average acceleration in the time interval $1 \leq t \leq 4$?

7. Show that if f is linear then

$$[\text{average value of } f \text{ on } p \leq x \leq q] = \frac{f(p) + f(q)}{2}.$$

9-3. Volumes of Solids of Revolution

The Fundamental Theorem of Calculus enables one to calculate areas by finding antiderivatives. Such techniques can be extended to enable calculation of arc length of curves and volumes and surface areas of solid figures to be made. A full treatment of these topics will be left to subsequent courses. In order to give you an indication of the wide use of antidifferentiation techniques we shall discuss in this section the problem of finding volumes of solids of revolution.

Suppose the region bounded by $y = f(x)$, $x = a$, $x = b$ and $y = 0$ is revolved about the x-axis, as shown in Figure 9-3a and b, obtaining a solid of revolution, as shown in Figure 9-3c.

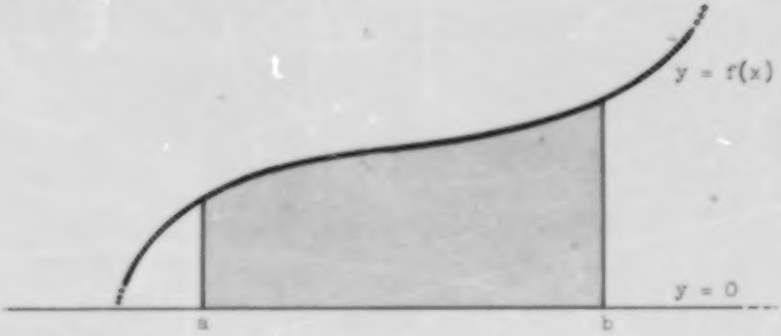


Figure 9-3a

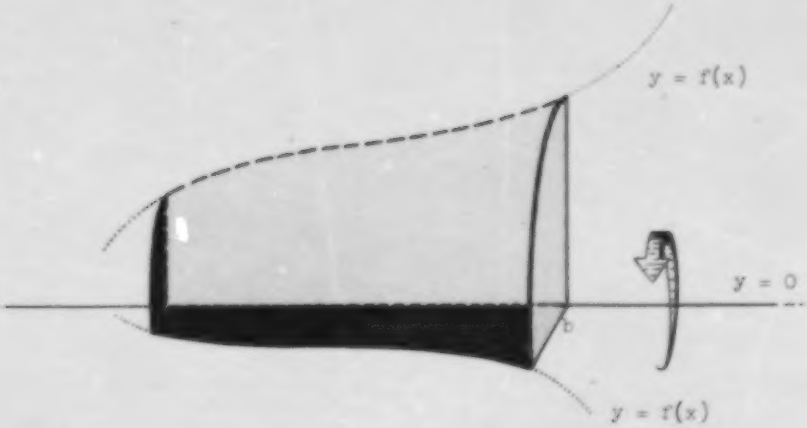


Figure 9-3b

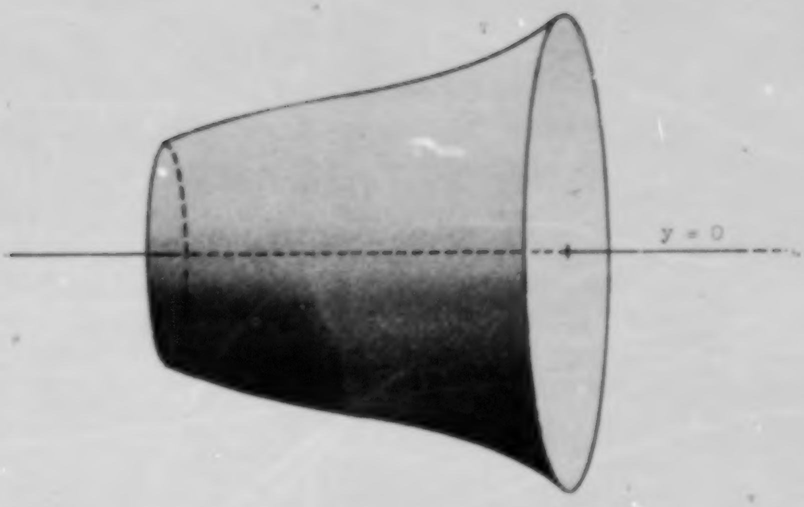


Figure 9-3c

The volume of this solid can be determined by a procedure similar to the one used to establish the Area Theorem. Let $V(t)$ be the volume of the solid obtained by revolving the region bounded by

$$y = f(x), \quad x = a, \; x = t \text{ and } y = 0$$

about the x-axis. ($V(t)$ is the volume of the shaded portion of Figure 9-3d.)

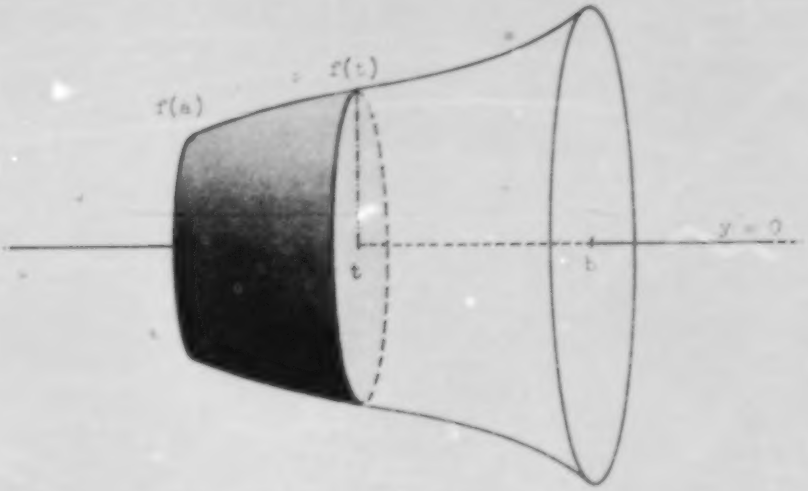


Figure 9-3d

This defines the volume function

$$V : t \to V(t), \quad a \le t \le b.$$

Let us also assume that f is increasing on the interval $a \le x \le b$ and that f is continuous and nonnegative at each point of this interval. By using elementary properties of volume we shall show that the derivative V' of V is given by

$$V'(t) = \pi(f(t))^2, \quad a \le t \le b. \tag{1}$$

The Fundamental Theorem of Calculus will then give us

$$V(b) - V(a) = \int_a^b \pi(f(t))^2\, dt$$

for V is a function whose derivative is $t \to \pi(f(t))^2$. Since $V(a) = 0$, we obtain from this the desired volume formula:

$$\boxed{V(b) = \int_a^b \pi(f(t))^2\, dt} \tag{2}$$

Let us now prove (1). Suppose t is fixed and that $h > 0$. The quantity

$$V(t + h) - V(t) \tag{3}$$

is the volume of the shaded region shown in Figure 9-3e.

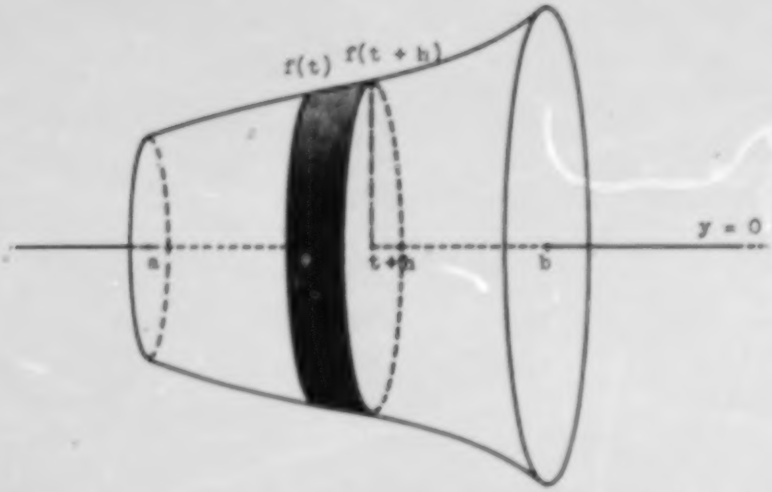


Figure 9-3e

$V(t + h) - V(t)$ = volume of shaded region

The function f is assumed to be increasing so that

$$f(t) \leq f(x) \leq f(t + h) \quad \text{for} \quad t \leq x \leq t + h.$$

Hence, the shaded solid in Figure 9-3e is included in the cylinder C_1 centered on the x-axis with radius $f(t + h)$ and length h. (See Figure 9-3f.) Furthermore, the shaded solid of

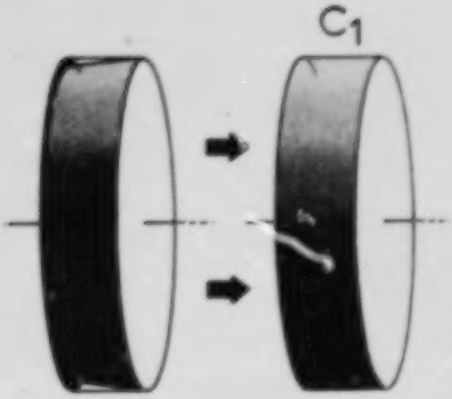


Figure 9-3f

Exterior Cylinder C_1

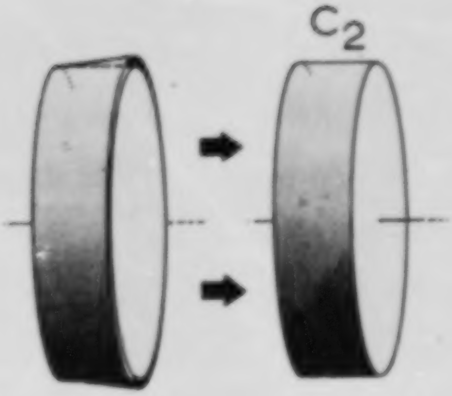


Figure 9-3g

Interior Cylinder C_2

Figure 9-3e includes the cylinder C_2 centered on the x-axis with radius $f(t)$ and length h. (See Figure 9-3g.) Recalling that the volume of a cylinder is

$$\pi \times (\text{radius})^2 \times \text{length}$$

we have

$$\text{volume } C_1 = \pi(f(t + h))^2 h$$

and

$$\text{volume } C_2 = \pi(f(t))^2 h.$$

Since the shaded region of Figure 9-3e has volume $V(t + h) - V(t)$, includes C_2 and is included in C_1, we have

$$\text{volume } C_2 \leq V(t + h) - V(t) \leq \text{volume } C_2$$

that is

$$\pi(f(t))^2h \le V(t+h) - V(t) \le \pi(f(t+h))^2h.$$

As h was assumed to be positive we can divide through by h to obtain

$$(4) \qquad \pi(f(t))^2 \le \frac{V(t+h) - V(t)}{h} \le \pi(f(t+h))^2,$$

As h approaches 0, $f(t+h)$ approaches $f(t)$ so that

$$(5) \qquad \frac{V(t+h) - V(t)}{h} \text{ approaches } \pi(f(t))^2, \text{ as } h \text{ approaches } 0.$$

If h is taken to be negative, the inequality (4) will be reversed but the conclusion (5) remains the same. This establishes that, indeed,

$$V'(t) = \pi(f(t))^2$$

and completes the proof of (1).

Remark: The same result (2) will hold if f is assumed to be decreasing or if it is assumed that the interval can be subdivided into subintervals so that on each subinterval f is always increasing or always decreasing (see the Remark in Section 7-3 after the proof of the Fundamental Theorem). The result can also be established using only the assumption that f is continuous.

Before examining some examples, let us interpret the formula (2) in terms of the concept of average value. Consider a cross-section of the solid of Figure 9-3d, perpendicular to the x-axis, cutting the x-axis at $(t,0)$, such as the shaded region R of Figure 9-3h.

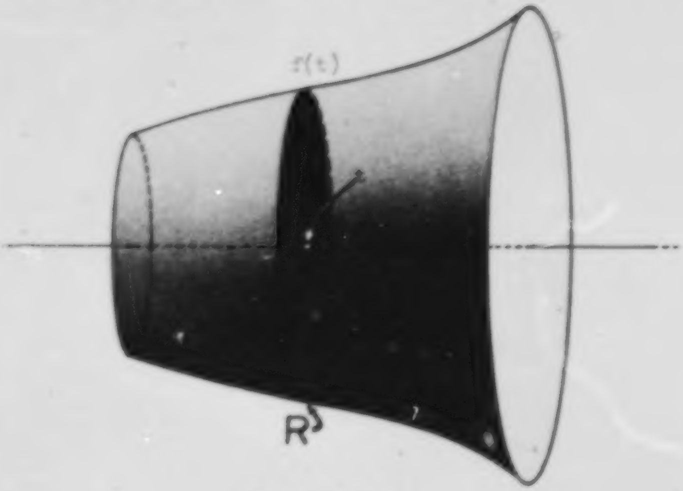


Figure 9-3h

A cross-section R.

The region R is circular and has radius $f(t)$ so its area is $\pi(f(t))^2$. For each t, $a \leq t \leq b$, the area $C(t)$ of the cross-section through $(t,0)$ therefore has area

$$C(t) = \pi(f(t))^2.$$

This defines the cross-sectional area function C,

$$C : t \rightarrow C(t) = \pi(f(t))^2.$$

The average value of C on the interval $a \leq t \leq b$ is

$$\frac{1}{b - a}\int_a^b C(t)dt = \frac{1}{b - a}\int_a^b \pi(f(t))^2\, dt$$

The volume of the solid of Figure 9-3d is thus

$$(b - a) \times \frac{1}{b - a}\int_a^b C(t)dt = \int_a^b \pi(f(t))^2 dt,$$

that is

(6) volume = (length) × (average cross-sectional area).

In other words the cylinder formula

$$\text{volume} = \pi r^2 h = (\text{cross-sectional area}) \times (\text{length})$$

can be extended to give the volume of a solid of revolution merely by replacing the cross-sectional area (which is constant for a cylinder) by the average cross-sectional area. This gives a convenient device for reconstructing the formula (2).

Example 9-3a. Find the volume of the solid of revolution obtained by revolving the region bounded by

$$y = \sin x, \quad x = 0, \quad x = \pi, \quad y = 0$$

about the x-axis.

To find the cross-sectional area function C, let R be a cross-section perpendicular to the x-axis through $(t,0)$. (See Figure 9-3i.) The region R is a circle with radius $\sin t$. Thus

$$C : t \rightarrow \pi \sin^2 t.$$

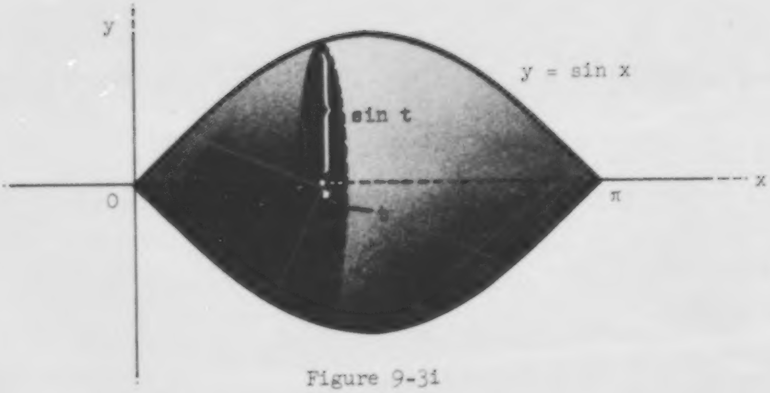


Figure 9-3i

The average value of C on the interval $0 \le t \le \pi$ is

$$\frac{1}{\pi - 0}\int_0^{\pi} C(t)dt = \frac{1}{\pi}\int_0^{\pi} \pi \sin^2 t\, dt$$

so the desired volume is

$$\begin{aligned}(\text{length}) \times (\text{average cross-sectional area}) &= \pi \cdot \frac{1}{\pi}\int_0^{\pi} \pi \sin^2 t\, dt \\ &= \pi \int_0^{\pi} \sin^2 t\, dt.\end{aligned}$$

To calculate this integral one can use the tables or recall that

$$\sin^2 t = \frac{1 - \cos 2t}{2}$$

so that

$$\begin{aligned}\pi \int_0^{\pi} \sin^2 t\, dt &= \frac{\pi}{2}\int_0^{\pi} (1 - \cos 2t)dt \\ &= \frac{\pi}{2}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi} \\ &= \frac{\pi^2}{2}\end{aligned}$$

which is our desired volume.

Example 9-3b. Find the volume of the solid of revolution obtained by revolving the region bounded by $x = 0$, $x = 2$, $y = 0$, $y = x^2$ about

(i) the x-axis

(ii) the y-axis.

In each case we shall find the cross-sectional area function and apply (6).

(i) Revolution about the x-axis

A cross-section perpendicular to the x-axis at $(t,0)$ has radius t^2 (see Figure 9-3j) and hence, the cross-sectional area function is:

$$C : t \to \pi t^4 .$$

The desired volume is

$$\text{(length)} \times \text{(average cross-sectional area)} = 2 \times \frac{1}{2-0}\int_0^2 C(t)dt$$

$$= \pi \int_0^2 t^4\, dt = \frac{32\pi}{5} .$$

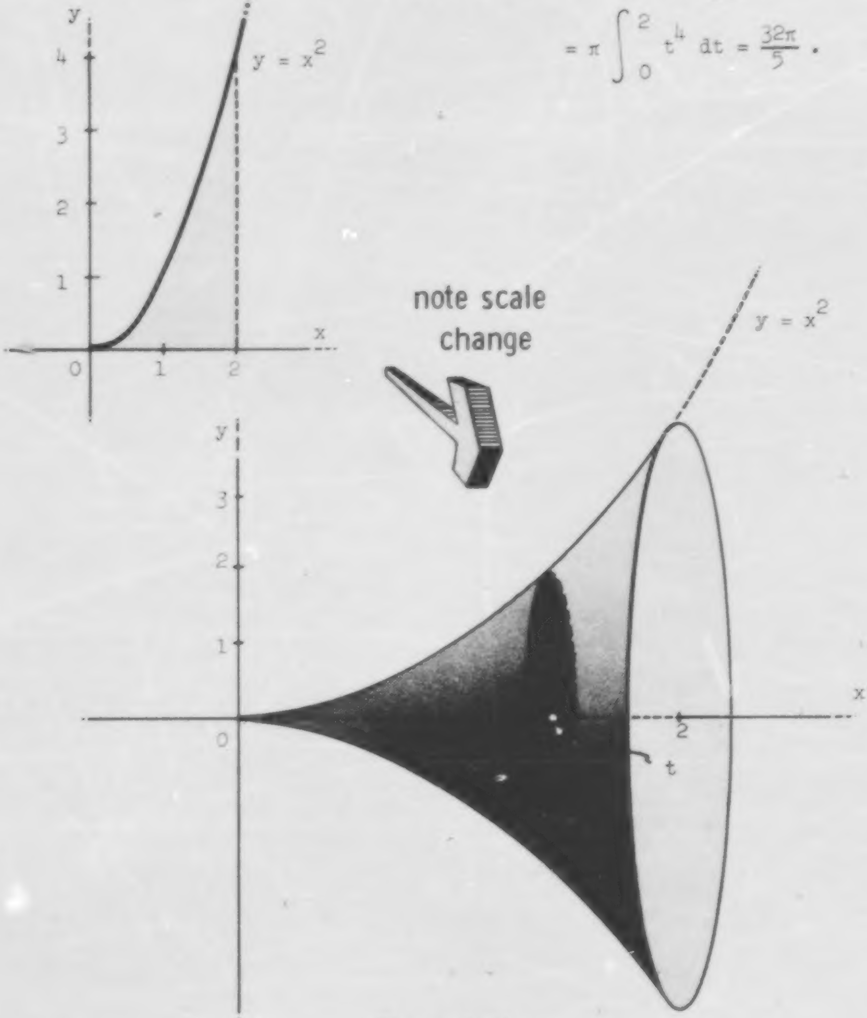

y
4
3
2
1
y = x^2
x
0
1
2
note scale change
y = x^2
y
3
2
1
x
0
2
t

Figure 9-3j

(ii) Revolution about the y-axis

Revolution of the region about the y-axis gives the hollowed out cylinder indicated [illegible] Figure 9-3k.

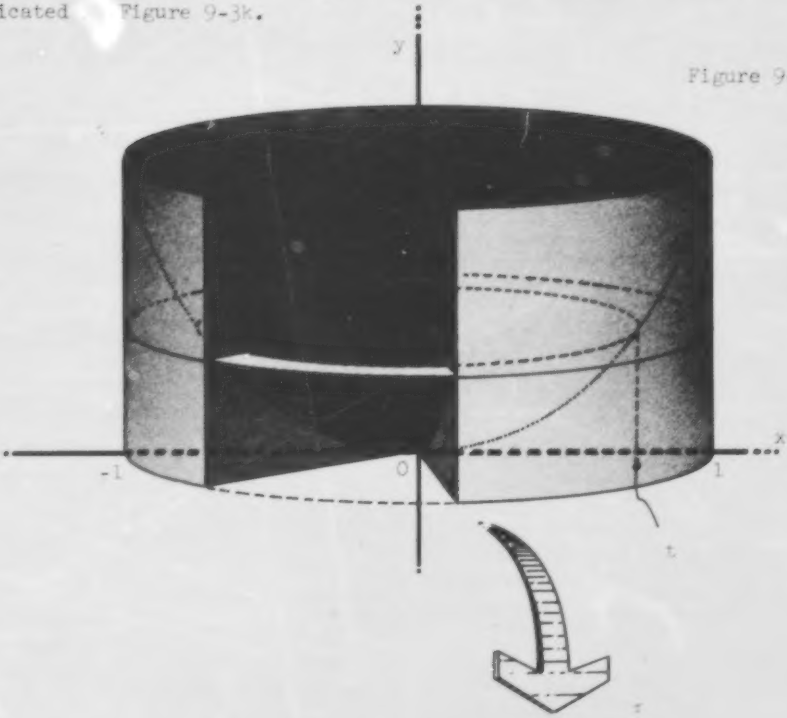


Figure 9-3k

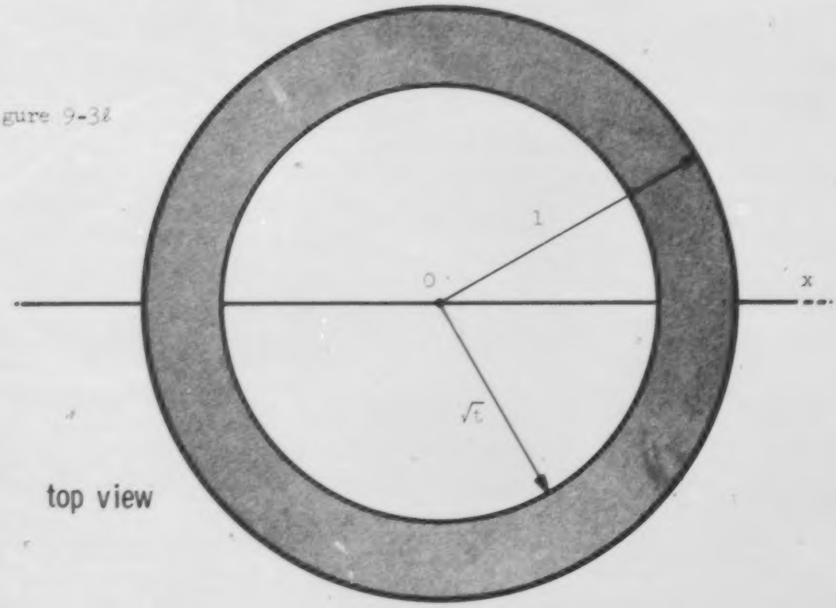

1
0
x
$\sqrt{t}$
top view

Figure 9-3ℓ

In this case we take cross-sections perpendicular to the y-axis through $(0,t)$. The cross-section is a circular ring with inner radius $\sqrt{t}$ and outer radius 1. (See Figure 9-3k and 9-3ℓ). The area of this cross-section is

$$\pi\, 1^2 - \pi(\sqrt{t})^2 = \pi(1 - t) = C(t).$$

The average value of the area for $0 \leq t \leq 4$ is

$$\frac{1}{4 - 0}\int_0^4 C(t)dt = \frac{\pi}{4}\int_0^4 (1 - t)dt$$

$$= \frac{\pi}{4}\left(t - \frac{t^2}{2}\right)\Big|_0^4 = \frac{\pi}{2}\,.$$

Hence, the desired volume is

$$(\text{length}) \times (\text{average cross-sectional area}) = 4 \cdot \frac{\pi}{2} = 2\pi.$$

Exercises 9-3

[Note: It is essential in problems of this type for your solution to be accompanied by a sketch.]

1. Find the volume of the solid generated by revolving about the x-axis the region below the graph of each of the following functions and above the indicated interval.

   (a) $f : x \rightarrow 3x, \quad 0 \leq x \leq 2$

   (b) $f : x \rightarrow \sqrt{x}, \quad 0 \leq x \leq 1$

   (c) $f : x \rightarrow 2x^{3/2}, \quad 0 \leq x \leq 1$

   (d) $f : x \rightarrow |x|, \quad -1 \leq x \leq 2$

   (e) $f : x \rightarrow -(x - 1)^2 + 4, \quad -1 \leq x \leq 2$

   (f) $f : x \rightarrow \sqrt{\log_e x}, \quad 1 \leq x \leq 5$

   (g) $f : x \rightarrow \sqrt{9 - x^2}, \quad 0 \leq x \leq 3$

   (h) $f : x \rightarrow \tan x, \quad 0 \leq x \leq \frac{\pi}{3}$

2. Use the procedure of this section to find the volume of a right circular cone of altitude h and base of radius r.

3. Obtain the formula for the volume of a sphere of radius r by first showing that the sphere is a solid of revolution.

4. Find the volume of the ellipsoid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis. (Assume $a > b$.)

5. Find the volume of the segment of a sphere of radius r bounded by two parallel planes if the bases of the segment are at distance a and b from the center and are on the same side.

6. Find the volume of the solid obtained by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about the x-axis.

7. A cylindrical hole of radius 1 inch is drilled out along a diameter of a solid sphere of radius 4 inches. Find the volume of the material cut out.

8. Find the volume of the portion of a sphere of radius r remaining after a cylindrical hole is drilled out along its diameter if the length of the hole is $2h$. Check your answer by considering some special cases.

9. Find the volume of the solid of revolution obtained by revolving the region bounded by $x = 0$, $x = 2$, $y = 0$, $y = x^2$ about

(a) the line $y = 4$.

(b) the line $y = -2$.

(c) the line $x = 2$.

(d) the line $x = 4$.

9-4. Estimation of Definite Integrals

The Fundamental Theorem of Calculus tells us that if f is continuous then the integral $\int_a^b f$ is $F(b) - F(a)$, if F is an antiderivative of f. Thus the problem of calculating areas (or average values, volumes of solids of revolution, etc.) can be solved if we can find an antiderivative for f. The problem of finding such an antiderivative in terms of elementary functions may not be solvable (see Example 9-1ℓ). Even if the problem is solvable the form of the antiderivative may be inconvenient. Various methods have been developed for estimating the integral $\int_a^b f$. With the advent of high-speed computers these methods have become valuable means for obtaining approximate solutions to area and related problems. Two such methods will be discussed in this section.

Let us suppose that f is increasing and continuous on the interval $a \leq x \leq b$ and that we seek to estimate $\int_a^b f$. The region bounded by f will be contained in the rectangle ABCD and will contain the rectangle ABC'D' (see Figure 9-4a), so that

$$\text{area } ABC'D' \leq \int_a^b f \leq \text{area } ABCD$$

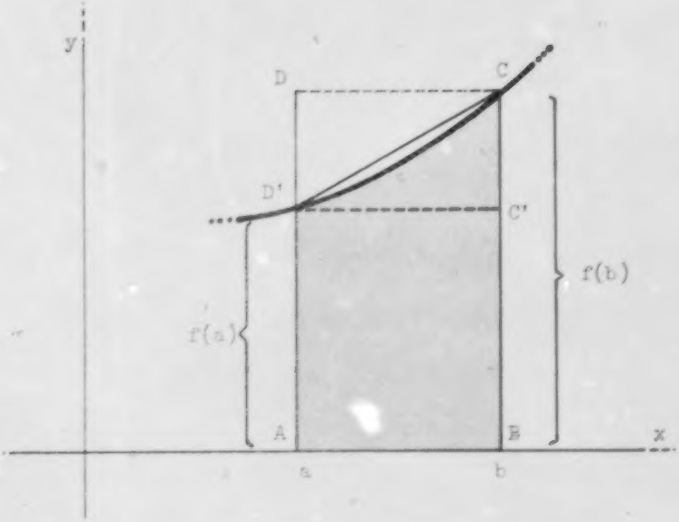


Figure 9-4a

The region ABCD has height $f(b)$ and length $b - a$ while the region ABC'D' has height $f(a)$ and length $b - a$, so that

$$f(a)(b - a) \leq \int_a^b f \leq f(b)(b - a).$$

If we take the average of the numbers

$$\alpha = f(a)(b - a) \quad \text{and} \quad \beta = f(b)(b - a)$$

we expect that this will give a better approximation to $\int_a^b f$ than either of the numbers α and β. This average is

$$\frac{\alpha + \beta}{2} = \frac{f(a) + f(b)}{2}(b - a).$$

This leads, therefore, to the approximation

$$(1) \qquad \int_a^b f \approx \frac{f(a) + f(b)}{2}(b - a).$$

This approximation will, in general, not be very good. It is, however, exact if f is linear, for if

$$f : x \to cx + d$$

then

$$\int_a^b f = \frac{c(b^2 - a^2)}{2} + d(b - a) = (b - a)\left[\frac{ca + d + cb + d}{2}\right]$$
$$= \frac{f(a) + f(b)}{2}(b - a).$$

The estimate (1) is just the area of the trapezoid ABCD of Figure 9-4a, that is, the estimate is the integral of the linear function obtained by connecting $(a,f(a))$ to $(b,f(b))$ with a straight line.

The estimate (1) does not, of course, require that f be increasing and can be used for more general functions. To obtain better approximations to the integral $\int_a^b f$ we can subdivide the interval $a \leq x \leq b$ into small subintervals, calculate the average (1) in each of these subintervals and add these together. Let us find a formula for this approximation in the case of equal subdivisions. Suppose n is a positive integer, and let the points $a_1, a_2, \ldots, a_{n-1}$ divide the interval $[a,b]$ into n equal sub-intervals each of length $\frac{b - a}{n}$, as shown in Figure 9-4b.

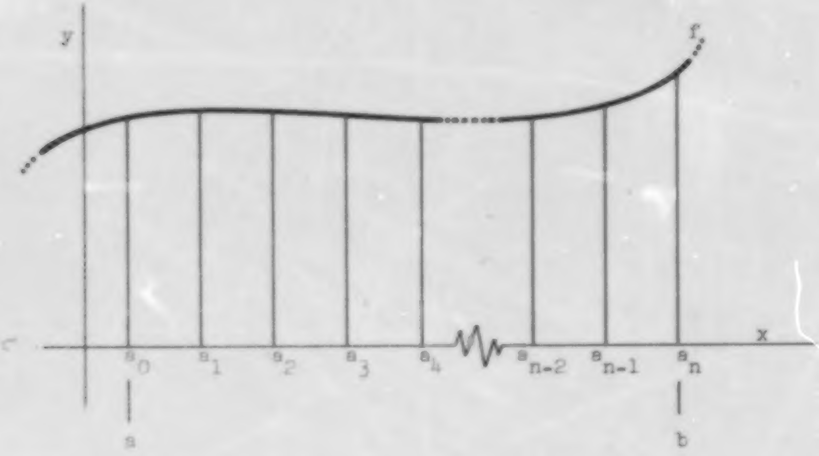


Figure 9-4b

Then we have

$$\int_a^b f = \int_{a_0}^{a_1} f + \int_{a_1}^{a_2} f + \int_{a_2}^{a_3} f + \ldots + \int_{a_{n-1}}^{a_n} f.$$

Each of the integrals $\int_{a_i}^{a_{i+1}} f$ is then approximated using (1), that is

$$\int_{a_i}^{a_{i+1}} f \approx \frac{f(a_i) + f(a_{i+1})}{2} (a_{i+1} - a_i), \; i = 0, 1, 2, \ldots, n - 1.$$

By adding these estimates we obtain

$$\int_a^b f \approx \frac{f(a_0) + f(a_1)}{2} (a_1 - a_0) + \frac{f(a_2) + f(a_1)}{2} (a_2 - a_1)$$
$$+ \frac{f(a_3) + f(a_2)}{2} (a_3 - a_2) + \ldots + \frac{f(a_{n-1}) + f(a_n)}{2} (a_n - a_{n-1})$$

Since each subinterval $a_i \leq x \leq a_{i+1}$ has length $\frac{b - a}{n}$ it follows that

$$a_{i+1} - a_i = \frac{b - a}{n}, \; i = 0, 1, 2, \ldots, n - 1.$$

Thus we can factor out $\frac{b - a}{2n}$ and obtain

$$\int_a^b f \approx \frac{b-a}{2n}((f(a_0) + f(a_1)) + (f(a_1) + f(a_2)) + (f(a_2) + f(a_3)) + \dots + (f(a_{n-1}) + f(a_n))).$$

The terms $f(a_0)$ and $f(a_n)$ appear once, while each of the terms $f(a_1)$, $f(a_2)$, ..., $f(a_{n-1})$ appears twice, that is:

$$(3) \quad \int_a^b f \approx \frac{b-a}{2n}(f(a_0) + 2f(a_1) + 2f(a_2) + \dots + 2f(a_{n-1}) + f(a_n)).$$

This approximation formula is known as the Trapezoidal Rule. It approximates the integral by the sum of the areas of the trapezoids obtained by connecting $(a_i, f(a_i))$ to $(a_{i+1}, f(a_{i+1}))$ by straight lines, as shown in Figure 9-4c.

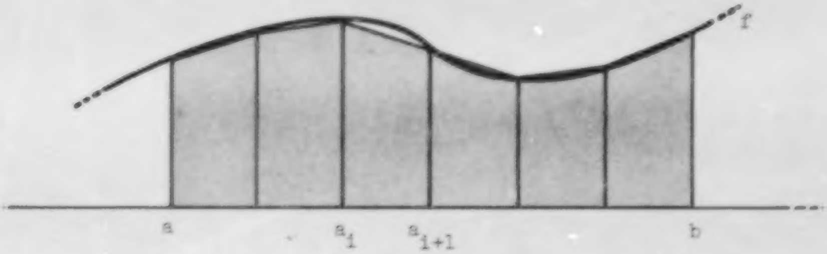


Figure 9-4c

An obvious question is, "How much error is involved in using

$$\frac{b-a}{2n}(f(a_0) + 2f(a_1) + 2f(a_2) + \dots + 2f(a_{n-1}) + f(a_n))$$

to approximate $\int_a^b f$?" It can be shown* that the error is at most

$$(4) \qquad \frac{M(b-a)^3}{12n^2}$$

where M is a bound for the second derivative on the interval, that is, $|f''(x)| \le M$, $a \le x \le b$.

---

*See Calculus, SMSG, p. 831

Example 9-4a. Estimate $\log_e 2 = \int_1^2 \frac{1}{x}\,dx$ correct to one decimal place be using the Trapezoidal Rule.

First divide the interval $1 \le x \le 2$ into two equal parts by setting

$$a_0 = 1,\ a_1 = \frac{3}{2},\ a_2 = 2.$$

The formula (3) gives (with $f : x \to \frac{1}{x}$, $n = 2$):

$$\log_e 2 = \int_1^2 \frac{1}{x}\,dx \doteq \frac{2-1}{2 \times 2}\left[1 + 2\left(\frac{2}{3}\right) + \frac{1}{2}\right] = \frac{17}{24}.$$

The first derivative of f is $f': x \to -\frac{1}{x^2}$ so the second derivative is $f'': x \to \frac{2}{x^3}$. This function is decreasing on the interval $1 \le x \le 2$ so its maximum on the interval is: $f''(1) = 2$. Using (4) with $b = 2$, $a = 1$, $n = 2$ and $M = 2$ the maximum error in the estimate $\log_e 2 \doteq \frac{17}{24}$ is

$$\frac{M(b-a)^3}{12n^2} = \frac{2 \cdot 1^3}{12 \cdot 4} = \frac{1}{24}.$$

In other words,

$$\frac{16}{24} \le \log_e 2 \le \frac{18}{24}.$$

Since $\frac{16}{24} > 0.66$ and $\frac{18}{24} = 0.75$, this tells us that

$$0.66 \le \log_e 2 \le 0.75.$$

Rounded off to one decimal place $\log_e 2$ could therefore, be either 0.7 or 0.8 so we need to choose n larger to obtain assurance as to the first decimal place in $\log_e 2$.

Let us try $n = 3$, which gives the points

$$a_0 = 1,\ a_1 = \frac{4}{3},\ a_2 = \frac{5}{3},\ a_3 = 2$$

and the estimate

$$\log_e 2 = \int_1^2 \frac{1}{x}\,dx \doteq \frac{2-1}{2 \times 3}\left[1 + 2\left(\frac{3}{4}\right) + 2\left(\frac{3}{5}\right) + \frac{1}{2}\right]$$
$$= \frac{21}{30}.$$

The maximum error is obtained from (4), with $b = 2$, $a = 1$, $n = 3$ and $M = 2$:

$$\frac{M(b-a)^3}{12n^2} = \frac{2 \cdot 1^3}{12 \cdot 9} = \frac{1}{54}$$

so that

$$\frac{21}{30} - \frac{1}{54} \leq \log_e 2 \leq \frac{21}{30} + \frac{1}{54} .$$

Since $\frac{21}{30} - \frac{1}{54} = \frac{184}{270} > 0.68$ and $\frac{21}{30} + \frac{1}{54} = \frac{194}{270} < 0.72$ we have

$$0.68 < \log_e 2 < 0.72$$

that is, correct to one decimal place, $\log_e 2 = 0.7$.

Simpson's Rule

Consider the Trapezoidal Rule in the case when $n = 2$:

$$\int_a^b f \approx \frac{b-a}{4} [f(a) + 2f(\frac{a+b}{2}) + f(b)].$$

Divide through by $b - a$ and write with denominator 3:

$$(5) \qquad \frac{1}{b-a}\int_a^b f \approx \frac{\frac{3}{4} f(a) + \frac{6}{4} f(\frac{a+b}{2}) + \frac{3}{4} f(b)}{3} .$$

This relation expresses the average value of f over the interval $a \leq x \leq b$ as a weighted average of the values of f at the endpoints and midpoint of the interval, the weights being $\frac{3}{4}, \frac{6}{4}, \frac{3}{4}$. This approximation is exact if f is linear on the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$, but it is not necessarily exact for non-linear functions. This raises the possibility that some other choice of weights might give a better estimate of the average value of f. In other words, we may be able to choose α_1, α_2, and α_3 so that

$$\alpha_1 + \alpha_2 + \alpha_3 = 3$$

and

$$\frac{\alpha_1 f(a) + \alpha_2 f(\frac{a+b}{2}) + \alpha_3 f(b)}{3}$$

is a better approximation to the average value of f on the interval than is (5).

Let us see if we can choose weights $\alpha_1, \alpha_2, \alpha_3$ so that $\alpha_1 + \alpha_2 + \alpha_3 = 3$ and the approximation

$$(6) \qquad \frac{1}{b-a}\int_a^b f \approx \frac{\alpha_1 f(a) + \alpha_2 f(\frac{a+b}{2}) + \alpha_3 f(b)}{3}$$

is *exact* *if* f *is* *a* *quadratic* *function*, say

$$f : x \to cx^2 + dx + e.$$

In this case

$$\int_a^b f = \frac{cx^3}{3} + \frac{dx^2}{2} + ex \Big|_a^b$$

$$= \frac{c}{3}(b^3 - a^3) + \frac{d}{2}(b^2 - a^2) + e(b - a)$$

$$= (b-a)[\frac{c}{3}(b^2 + ab + a^2) + \frac{d}{2}(b + a) + e]$$

$$= \frac{b-a}{6}[(ca^2 + da + e) + 4(c(\frac{a+b}{2})^2 + d(\frac{a+b}{2}) + e) + (cb^2 + db + e)]$$

$$= \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)].$$

Thus, when f is a quadratic:

$$\frac{1}{b-a}\int_a^b f = \frac{\frac{1}{2}f(a) + 2f(\frac{a+b}{2}) + \frac{1}{2}f(b)}{3}.$$

In other words, the choice of weights $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 2$, $\alpha_3 = \frac{1}{2}$ will make the approximation (6) *exact* for quadratic functions. The resulting approximation

$$(7) \qquad \int_a^b f \approx \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

is known as *Simpson's Rule*.

The approximation can be improved by subdividing the interval $a \le x \le b$ into small subintervals, calculating the weighted average (7) in each subinterval and adding these together. To obtain the general formula in the case of equal subintervals, suppose n is a positive integer. Let the points a_0, $a_2, a_4, \ldots, a_{2n}$ be the endpoints of n equal subintervals of $[a,b]$, each of length $\frac{b-a}{n}$, and let the points $a_1, a_3, a_5, \ldots, a_{2n-1}$ be the respective midpoints. (See Figure 9-4d.)

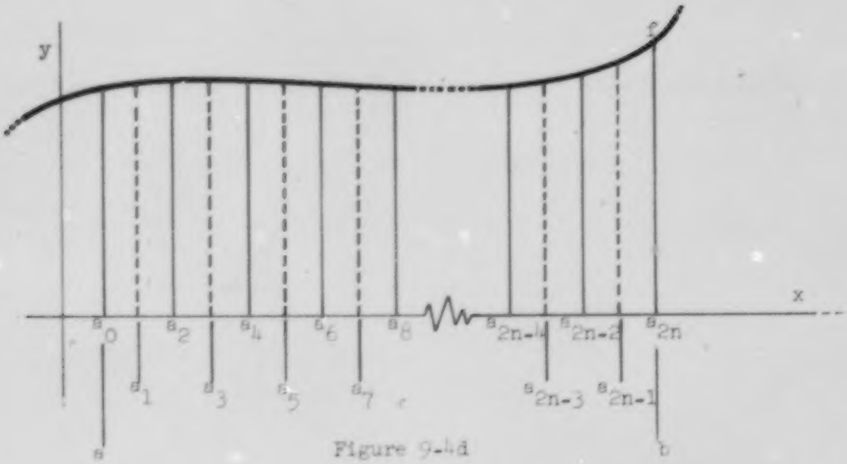


Figure 9-4d

We have:

$$\int_a^b f = \int_{a_0}^{a_2} f + \int_{a_2}^{a_4} f + \ldots + \int_{a_{2n-2}}^{a_{2n}} f.$$

Each of the integrals $\int_{a_{2i}}^{a_{2i+2}} f$ is then approximated using (7), i.e.,

$$\int_{a_{2i}}^{a_{2i+2}} f \approx \frac{b - a}{6n}(f(a_{2i}) + 4f(a_{2i+1}) + f(a_{2i+2})),\ i = 0, 1, 2, \ldots, n - 1.$$

Add these together for $i = 0, 1, 2, \ldots, n - 1$ and factor out $\frac{b - a}{6n}$:

$$\int_a^b f \approx \frac{b - a}{6n}[(f(a_0) + 4f(a_1) + f(a_2)) + (f(a_2) + 4f(a_3) + f(a_4))$$
$$+ (f(a_4) + 4f(a_5) + f(a_6)) + \ldots + (f(a_{2n-2})$$
$$+ 4f(a_{2n-1}) + f(a_{2n}))]$$

which can be regrouped to give

$$(8) \qquad \int_a^b f \approx \frac{b - a}{6n}[f(a_0) + 4f(a_1) + 2f(a_2) + 4f(a_3) + 2f(a_4)$$
$$+ \ldots + 2f(a_{2n-2}) + 4f(a_{2n-1}) + f(a_{2n})].$$

Note that the coefficients of $f(a_0)$ and $f(a_{2n})$ are each 1, the coefficients of the remaining endpoint values $f(a_2), f(a_4), \ldots, f(a_{2n-2})$ are each 2 and the coefficients of the midpoint values $f(a_1), f(a_3), \ldots,$

$f(a_{2n-1})$ are each 4. The number $2n + 1$ is the number of points used in the estimate. The formula (8) is the general form of *Simpson's Rule*.

It can be shown* that the error in using *Simpson's Rule* is at most

$$\frac{M_1(b - a)^5}{180(2n)^4} \tag{9}$$

where M_1 is a bound for the *fourth* derivative $f^{(4)}$ on the interval, that is $|f^{(4)}(x)| \leq M_1$, $a \leq x \leq b$. Comparing this with the trapezoidal error (4): $\frac{M(b - a)^3}{12n^2}$ we see that if n is large enough the factor $\frac{1}{n^4}$ in (9) is much smaller than $\frac{1}{n^2}$ in (4) and hence, the approximation using Simpson's Rule will usually be better than the approximation using the Trapezoidal Rule.

Example 9-4b. Use Simpson's Rule with $n = 1$ and $n = 2$ to estimate

$$\log_e 2 = \int_1^2 \frac{1}{x}\,dx.$$

With $n = 1$, we have

$$a_0 = 1,\ a_1 = \frac{1 + 2}{2} = \frac{3}{2},\ a_2 = 2$$

and (7) (which is (8) when $n = 1$) gives

$$\log_e 2 = \int_1^2 \frac{1}{x}\,dx \approx \frac{2 - 1}{6}[1 + 4(\frac{2}{3}) + \frac{1}{2}]$$

$$= \frac{1}{6}(\frac{25}{6}) = \frac{25}{36}.$$

The derivatives of $f : x \rightarrow \frac{1}{x}$ are

$$f' : x \rightarrow -\frac{1}{x^2};\quad f'' : x \rightarrow \frac{2}{x^3};$$

$$f''' : x \rightarrow \frac{-6}{x^4};\quad f^{(4)} : x \rightarrow \frac{24}{x^5}.$$

---

[*] See *Calculus*, SMSG, pp. 833-4.

The function $f^{(4)}$ is decreasing on the interval $1 \leq x \leq 2$ so its maximum is $f^{(4)}(1) = 24$. Now use the error estimate (9) with $b = 2$, $a = 1$, $n = 1$, $M_1 = 24$ to obtain the maximum error

$$\frac{24(2 - 1)^5}{180 \cdot 2^4} = \frac{1}{120} .$$

Thus, we know that

$$\frac{25}{36} - \frac{1}{120} \leq \log_e 2 \leq \frac{25}{36} + \frac{1}{120} .$$

Calculation gives:

$$\frac{25}{36} - \frac{1}{120} = \frac{496}{720} > 0.688$$

$$\frac{25}{36} + \frac{1}{120} = \frac{506}{720} < 0.703$$

so that

$$0.688 < \log_e 2 < 0.703$$

so that, correct to one decimal place $\log_e 2 = 0.7$.

Notice that by using the values of f at 3 points and Simpson's Rule we obtained one decimal place accuracy, while the Trapezoidal Rule would not guarantee this for 3 points (see Example 9-4a).

The case $n = 2$ will substantially improve the accuracy, for the error estimate (9) (with $b = 2$, $a = 1$, $n = 2$, $M_1 = 24$) gives

$$\frac{24(2 - 1)^2}{180 \cdot (4)^4} = \frac{1}{1920} .$$

In this case

$$a_0 = 1$$

$$a_1 = \frac{5}{4}$$

$$a_2 = \frac{3}{2}$$

$$a_3 = \frac{7}{4}$$

$$a_4 = 2$$

and Simpson's Rule (8) gives

$$\log_e 2 = \int_1^2 \frac{1}{x}\,dx \approx \frac{2-1}{6 \times 2}\left[1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2}\right]$$

$$= \frac{1}{12}\left(\frac{1747}{210}\right) = \frac{1747}{2520}$$

so that

$$\frac{1747}{2520} - \frac{1}{1920} \leq \log_e 2 \leq \frac{1747}{2520} + \frac{1}{1920} .$$

Let us use the estimates

$$\frac{1}{1920} < .0006$$

$$0.6932 < \frac{1747}{2520} < 0.6933$$

from which we obtain

$$0.6926 < \log_e 2 < 0.6939.$$

Thus, correct to two decimal places $\log_e 2 = 0.69$. To obtain the same accuracy with the Trapezoidal Rule we need to use the value of f at 14 points of the interval!

Exercises 9-4

1. Estimate $\int_0^1 \frac{1}{1+t^2}\,dt$ using the Trapezoidal Rule with

   (a) $n = 2$ (i.e., three points)

   (b) $n = 4$

   Estimate the error in each case. Also, compare your result with the known value of $\int_0^1 \frac{1}{1+t^2}\,dt$.

2. Estimate $\int_0^2 e^{-x^2}\,dx$ using tables for e^{-x} and the Trapezoidal Rule with $n = 2$ and $n = 4$. Estimate the error in each case.

3. Estimate $\int_0^1 \frac{1}{1+t^2}\,dt$ using Simpson's Rule with

   (a) $n = 1$ (i.e., three points)

   (b) $n = 3$

   Estimate the error in each case and compare with the known value of the integral.

4. Estimate $\int_0^2 e^{-x^2}\,dx$ using Simpson's Rule with $n = 1$ (three points) and $n = 2$ (five points). Estimate the error in each case.

5. Show that Simpson's Rule is exact for cubics, that is

$$\int_a^b f = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

   if $f : x \to Ax^3 + Bx^2 + Cx + D$. (Hint: It is enough to establish this for the case $B = C = D = 0$ since it is known to be true for quadratics.)

6. Suppose f is a convex function. Will the Trapezoidal Rule give too large or too small an estimate for $\int_a^b f$?

7. The letter n appears in (3) and (8). How is each use of n related to the number of points used in the estimate?

8. How large should n be taken in Simpson's Rule to give $\frac{\pi}{4} = \int_0^1 \frac{1}{1+t^2}\,dt$ accurately to five decimal places?

9. Use either Simpson's Rule or the Trapezoidal Rule to estimate $\log_e 3$, correct to four decimal places. Use the Reciprocals Table to aid in computation.

9-5. Taylor Approximations

In (6) of Section 7-5 we noted that

$$(1) \qquad \int_a^b f(t)dt \le \int_a^b g(t)dt \quad \text{if} \quad f(t) \le g(t), \quad a \le t \le b.$$

This inequality can be used to obtain the Taylor approximations for a given function with remainder estimates.

We first illustrate this process for the exponential function $x \to e^x$ on the interval $[0,M]$, that is, for $0 \le x \le M$. On this interval

$$1 \le e^x.$$

In (1) we take $f(t) = 1$ and $g(t) = e^t$ with $a = 0$ and $b = x$. (See Figure 9-5a.)

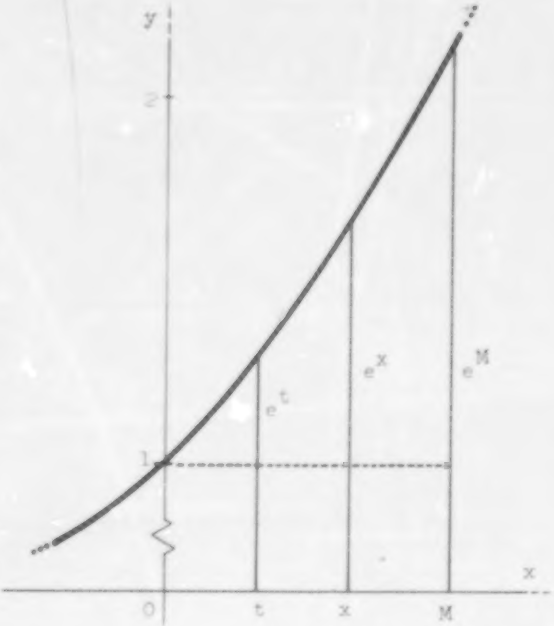


Figure 9-5a

Then

$$\int_0^x 1\ dt \le \int_0^x e^t\ dt.$$

Carrying out the integrations, we obtain

$$x \leq e^x - 1$$

or

$$1 + x \leq e^x.$$

Now we apply (1) to these resulting functions, again with $a = 0$ and $b = x$, and we obtain

$$\int_0^x (1 + t)dt \leq \int_0^x e^t\, dt.$$

(See Figure 9-5b.)

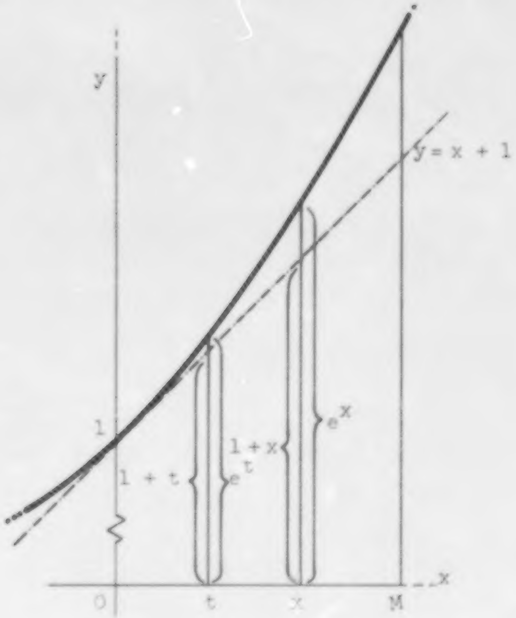


Figure 9-5b

Thus $$x + \frac{x^2}{2} \leq e^x - 1,$$

and $$1 + x + \frac{x^2}{2} \leq e^x.$$

Repeating this process, we obtain successively

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} \leq e^x.$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \leq e^x,$$

$$\cdots$$

$$(2) \qquad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \le e^x.$$

All of those approximations to e^x are too small. To obtain upper approximations, we use the fact that for $0 \le x \le M$,

$$e^x \le e^M$$

(recall that $f : x \to e^x$ is an increasing function).

We use (1) with $f(t) = e^t$, $g(t) = e^M$, and, as before, $a = 0$, $b = x$. (See Figure 9-5c.)

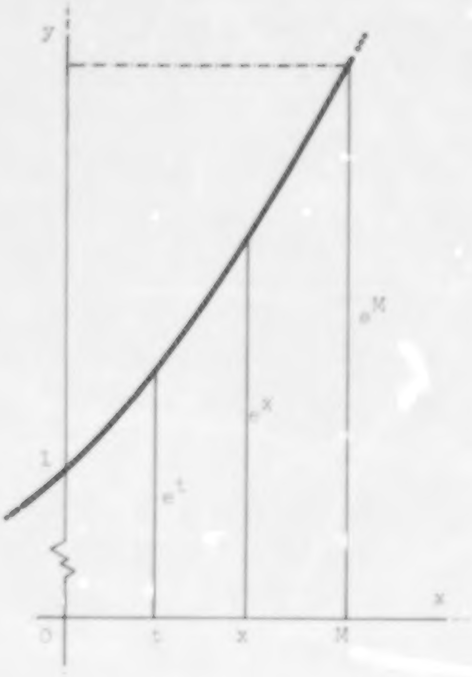


Figure 9-5c

We have

$$\int_0^x e^t\,dt \le \int_0^x e^M\,dt,$$

that is,

$$e^x - 1 \le e^M x$$

or

$$e^x \le 1 + e^M \cdot x.$$

With $f(t) = e^t$ and $g(t) = 1 + e^M \cdot t$, (1) gives (see Figure 9-5d)

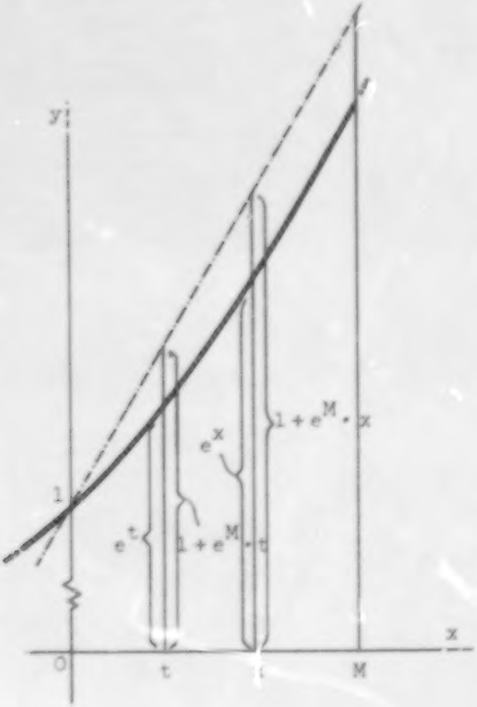

y
1
O
t
x
M
x
e^t
$1 + e^M \cdot t$
e^x
$1 + e^M \cdot x$

Figure 9-5d

$$\int_0^x e^t\,dt \leq \int_0^x (1 + e^M \cdot t)dt$$

or

$$e^x - 1 \leq x + e^M \frac{x^2}{2},$$

that is

$$e^x \leq 1 + x + e^M \frac{x^2}{2}.$$

If we continue in this fashion we have

(3) $$e^x \leq 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{e^M x^{n+1}}{(n+1)!}.$$

From (2) and (3)

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x)$$

where

$$0 \le R_n(x) \le \frac{e^M x^{n+1}}{(n+1)!} .$$

We used this result in Section 6-6 to estimate values of e^x where we chose the interval $[0,1]$ so that $M = 1$ and

$$R_n(x) \le \frac{e\, x^{n+1}}{(n+1)!} .$$

The procedure used with $f : x \to e^x$ can be applied to other functions. The essential idea is to start with information about the derivatives of the function on an interval, $[0,M]$, say. In the case of $f : x \to e^x$, all derivatives f', f'', f''', ..., are the same as f itself, so that to say, for example, that

$$\alpha \le f(x) \le \beta \quad \text{on } [0,M] \tag{4}$$

is the same as to say that

$$\alpha \le f^{(n)}(x) \le \beta \quad \text{on } [0,M]. \tag{5}$$

The generalization that we require is not (4) but (5).

To illustrate the general procedure let us take $n = 4$. We begin, therefore, with

$$\alpha \le f^{(4)}(x) \le \beta \quad \text{on } [0,M].$$

We work first with the left inequality

$$\alpha \le f^{(4)}(x)$$

and integrate from 0 to x, $(0 \le x \le M)$. Then

$$\int_0^x \alpha\, dt \le \int_0^x f^{(4)}(t)dt$$

and

$$\alpha x \le f'''(x) - f'''(0).$$

Hence,

$$f'''(0) + \alpha x \le f'''(x).$$

Integrate again from 0 to x

$$\int_0^x (f'''(0) + \alpha t)dt \le \int_0^x f'''(t)dt$$

to obtain

$$f'''(0)x + \frac{\alpha x^2}{2} \le f''(x) - f''(0)$$

and

$$f''(0) + f'''(0)x + \frac{\alpha x^2}{2} \le f''(x).$$

Continuing

$$f'(0) + f''(0)x + f''(0)\,\frac{x^2}{2!} + \frac{\alpha x^3}{3!} \le f'(x),$$

and finally,

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{\alpha x^4}{4!} \le f(x). \tag{6}$$

If we work with

$$f^{(4)}(x) \le \beta$$

in the same way we obtain

$$f(x) \le f(0) + f'(0)x + f''(0)\,\frac{x^2}{2!} + f'''(0)\,\frac{x^3}{3!} + \frac{\beta x^4}{4!}\,. \tag{7}$$

Hence, from (6) and (7)

$$f(x) = f(0) + f'(0)x + \ldots + f'''(0)\,\frac{x^3}{3!} + R_3(x) \tag{8}$$

where

$$\frac{\alpha x^4}{4!} < R_3(x) < \frac{\beta x^4}{4!}\,.$$

The polynomial

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

is the Taylor approximation to f of degree not exceeding three as it satisfies

$$p_3(0) = f(0),\ p_3'(0) = f'(0),\ p_3''(0) = f''(0),\ \text{and}\ p_3'''(0) = f'''(0).$$

The inequalities (6) and (7) can then be written as

$$\frac{\alpha x^4}{4!} \le f(x) - p_4(x) \le \frac{\beta x^4}{4!} \quad \text{for} \quad 0 \le x \le M.$$

In general, if

$$\alpha \le f^{(n+1)}(x) \le \beta, \quad 0 \le x \le M$$

and

$$p_n(x) = f(0) + f'(0) + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n)}(0)}{n!}x^n \tag{9}$$

then p_n is the nth Taylor approximant to f and

$$\frac{\alpha x^{n+1}}{(n+1)!} \le f(x) - p_n(x) \le \frac{\beta x^{n+1}}{(n+1)!}\ ; \ 0 \le x \le M. \tag{10}$$

For nonpositive x analogous results can be obtained. For example, if

$$|f^{n+1}(x)| \le K \quad \text{for} \quad 0 \le |x| \le M$$

and p_n is the Taylor approximation (14) then

$$|f(x) - p_n(x)| \le K\frac{|x|^{n+1}}{(n+1)!} \quad \text{for} \quad 0 \le |x| \le M. \tag{11}$$

Of course, all these results assume that $f^{(n+1)}$ satisfies the conditions of the Fundamental Theorem of Calculus (see Section 7-3).

Let us look at another example.

Example 9-5a. Find the third degree Taylor approximation to $f : x \to \sqrt{1 + x}$ and an error estimate for $0 \le x \le 1$.

Writing $f(x) = (1 + x)^{1/2}$ and using the power rule $(Du^{\alpha} = \alpha u^{\alpha-1}Du)$ with $u = 1 + x$ we obtain the successive derivatives:

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(1 + x)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(1 + x)^{-5/2}$$

$$f^{(4)}(x) = -\frac{15}{16}(1 + x)^{-7/2}.$$

In particular

$$f(0) = 1,\ f'(0) = \frac{1}{2},\ f''(0) = -\frac{1}{4},\ f'''(0) = \frac{3}{8},$$

so the third Taylor approximant to $f : x \to (1 + x)^{1/2}$ is

$$\begin{aligned} p_3(x) &= 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2!} + \frac{3}{8}\cdot\frac{x^3}{3!} \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3. \end{aligned} \tag{12}$$

According to (10) the error in $p_3(x)$ is determined by

$$f^{(4)}(x) = -\frac{15}{16}(1 + x)^{-7/2}.$$

Since f is decreasing on the interval $0 \leq x \leq 1$,

$$f^{(4)}(1) \geq f^{(4)}(x) \geq f^{(4)}(0).$$

Substituting we calculate

$$f^{(4)}(1) = -\frac{15}{16}(1 + 1)^{-7/2} = \frac{-15\sqrt{2}}{256}$$

and $$f^{(4)}(0) = -\frac{15}{16}(1 + 0)^{-7/2} = -\frac{15}{16},$$

so $$-\frac{15}{16} \leq f^{(4)}(x) \leq -\frac{15\sqrt{2}}{256}, \quad 0 \leq x \leq 1.$$

We conclude from (10) that

$$(13) \qquad -\frac{15}{16}\frac{x^4}{4!} \leq (1 + x)^{1/2} - p_3(x) \leq -\frac{15\sqrt{2}}{256}\frac{x^4}{4!}, \quad 0 \leq x \leq 1.$$

In particular, if $x = 0.2$

$$p_3(0.2) = 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3$$
$$= 1.0955,$$

which indicates that $\sqrt{1.2}$ is approximately 1.0955. The error in this approximation may be found by substituting $x = 0.2$ in (13):

$$-\frac{15}{16}\cdot\frac{(0.2)^4}{4!} \leq \sqrt{1.2} - p(0.2) \leq -\frac{15\sqrt{2}}{256}\cdot\frac{(0.2)^4}{4!}$$

which works out to be

$$-.00006 \leq \sqrt{1.2} - p(0.2) \leq -.000011$$

from which we conclude that

$$1.09544 \leq \sqrt{1.2} \leq 1.09549,$$

or $\sqrt{1.2} \approx 1.0954$, correct to 4 decimal places.

The Logarithm and Arctangent Functions

The above methods can be applied to give the Taylor approximations to

$$x \rightarrow \log_e(1 + x) \quad \text{and} \quad x \rightarrow \arctan x$$

with remainder estimates. These results can be obtained in sharper form by noting that

$$(14) \qquad \log_e(1 + x) = \int_0^x \frac{1}{1 + t}\, dt, \; x > -1$$

and

$$(15) \qquad \arctan x = \int_0^x \frac{1}{1 + t^2}\, dt$$

and finding suitable expressions for $\frac{1}{1 + t}$ and $\frac{1}{1 + t^2}$. From the formula for the sum of a geometric progression,

$$1 + x + x^2 + x^3 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} - \frac{x^n}{1 - x},$$

we obtain

$$(16) \qquad \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots + x^{n-1} + \frac{x^n}{1 - x}.$$

If $x = -t$, (16) becomes the desired expansion of $\frac{1}{1 + t}$;

$$(17) \qquad \frac{1}{1 + t} = 1 - t + t^2 - t^3 + \dots + (-1)^{n-1}t^{n-1} + (-1)^n \frac{t^n}{1 + t}.$$

If $x = -t^2$, (16) becomes the desired expansion of $\frac{1}{1 + t^2}$;

$$(18) \qquad \frac{1}{1 + t^2} = 1 - t^2 + t^4 - t^6 + \dots + (-1)^{n-1}t^{2n-2} + (-1)^n \frac{t^{2n}}{1 + t^2}.$$

Using (17), we obtain for $x > -1$,

$$\begin{aligned}
\log_e(1 + x) &= \int_0^x \frac{1}{1 + t}\, dt \\
&= \int_0^x (1 - t + t^2 - t^3 + \dots + (-1)^{n-1}t^{n-1})dt + (-1)^n \int_0^x \frac{t^n}{1 + t}\, dt \\
&= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots + (-1)^{n-1} \frac{t^n}{n} \Big|_0^x + (-1)^n \int_0^x \frac{t^n}{1 + t}\, dt \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1 + t}\, dt.
\end{aligned}$$

Using (18) we similarly obtain

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \int_0^x (1 - t^2 + t^4 - t^6 + \dots + (-1)^{n-1}t^{2n-2})dt + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^{n-1}\frac{x^{2n}}{2n} + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

We conclude that

$$(19) \qquad \boxed{\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + R_n}$$

where

$$(20) \qquad R_n = (-1)^n \int_0^x \frac{t^n}{1+t}\,dt,$$

and

$$(21) \qquad \boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^{n-1}\frac{x^{2n}}{2n} + R_n}$$

where

$$(22) \qquad R_n = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

In Section 6-9 it was shown that if

$$f : x \to \log_e(1+x)$$

and

$$p_n : x \to x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1}\frac{x^n}{n}$$

then

$$f(0) = p_n(0),\ f'(0) = p'_n(0),\ f''(0) = p''_n(0),\ \dots,\ f^{(n)}(0) = p_n^{(n)}(0)$$

so that p_n is the nth degree Taylor approximant to f. Hence, (19) and (20) give an explicit formula for the error R_n involved in using $p_n(x)$ to approximate $f(x)$. The size of R_n can be easily estimated from (20).

For example, if $0 \le x \le 1$, and we put

$$g : t \to \frac{t^n}{1+t},$$

then

$$g(t) \leq t^n \quad \text{if} \quad 0 \leq t \leq x$$

(since $1 + t \geq 1$ if $t \geq 0$) so that

$$\int_0^x g \leq \int_0^x t^n \, dt = \left. \frac{t^{n+1}}{n+1} \right|_0^x = \frac{x^{n+1}}{n+1}$$

and hence,

$$(23) \qquad |R_n| = \int_0^x \frac{t^n}{1+t} \, dt = \int_0^x g \leq \frac{x^{n+1}}{n+1} .$$

Therefore, the error in using $p_n(x)$ to approximate $\log_e(1 + x)$, for $0 \leq x \leq 1$ is at most $\frac{x^{n+1}}{n+1}$. This will be small if n is large.

Other intervals for x are considered in the exercises. In particular, it will be shown that if $x > 1$, then R_n will not approach 0 as n becomes large, but in fact

$$\lim_{n \to \infty} R_n = \infty \quad \text{if} \quad x > 1.$$

Hence, for $x > 1$, the approximations $p_n(x)$ differ substantially from $\log_e(1 + x)$ when n is large.

The methods are easily adapted to show that (21) and (22) give the Taylor approximations to the arctangent and an explicit formula for the error.

Example 9-5b. Use $n = 5$ in (19) to estimate $\log_e 2$.

Formula (19) gives

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + R_5$$

where

$$R_5 = -\int_0^1 \frac{t^5}{1+t} \, dt.$$

Using (23) we have

$$|R_5| \leq \frac{1}{6} .$$

Thus, the estimate

$$\log_e 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{6} + \frac{1}{5} \approx 0.783$$

is within $\frac{1}{6}$ of being correct. This is not very good, in fact, if we wish to use (19) to estimate $\log_e 2$ we must choose n very large to obtain much accuracy. Clearly, Simpson's Rule is a much more useful method for approximating values of $\log_e$.

Exercises 9-5

1. Start with the inequality $-1 \leq \cos x \leq 1$, and by repeated integration from 0 to x, $x \geq 0$ obtain

   (a) $-x \leq \sin x \leq x$

   (b) $-\frac{x^2}{2} \leq 1 - \cos x \leq \frac{x^2}{2}$

   (c) $-\frac{x^3}{3!} \leq x - \sin x \leq \frac{x^3}{3!}$

   (d) $-\frac{x^4}{4!} \leq \cos x - (1 - \frac{x^2}{2!}) \leq \frac{x^4}{4!}$

   (e) $-\frac{x^5}{5!} \leq \sin x - (x - \frac{x^3}{3!}) \leq \frac{x^5}{5!}$

2. Establish the inequalities of Number 1 for $x \leq 0$. (Hint: Rather than repeat the integrations use the odd and even function ideas.)

3. Find the third degree Taylor approximation to $x \to \sqrt[3]{1 + x}$ and an error estimate for $0 \leq x \leq 1$.

4. (a) Estimate the error in the third degree estimate for $x \to \sqrt{1 + x}$ in the interval $-1 < x \leq 0$.

   (b) Do the same for the interval $-.5 \leq x \leq 0$.

5. Consider the function $f : x \to \frac{1}{1 + x}$.

   (a) Show that the formula (17) gives the Taylor approximation to f. [I.e., $p_{n-1}(x) = 1 - x + x^2 \ldots (-1)^{n-1}x^{n-1}$ is the $(n - 1)^{st}$ Taylor approximation of f.]

   (b) Assume that the error $|R_{n-1}| \leq \frac{|x|^n}{|1 + x|}$. Find a statement for $|R_n|$.

   (c) If $x = 10$ what is the error using $p_5(10)$ to approximate $\frac{1}{11}$?

   (d) How does $\frac{1}{1 + x}$ differ from $p_{n-1}(x)$ if $x > 1$ and n is large?

6. Find the $p_{n-1}(x)$ Taylor approximation with an explicit remainder formula for $f : x \to \frac{1}{2 + x}$ [Hint: $\frac{1}{2 + x} = \frac{1}{2}\left(\frac{1}{1 + \frac{x}{2}}\right)$]. For what values of x will the remainder approach 0 as $n \to \infty$?

7. Do Number 6 for the function $f : x \to \log_e(2 + x)$.

8. Recall that

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} .$$

(a) Show that

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3} .$$

(b) Find π, correct to two decimal places by using (a) and formulas (21) and (22). How many terms do you need to use?

9. (a) Show that

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} .$$

(b) Use (a) to find π correct to two decimal places. How many terms do you need to use?

Using this method π has been claculated on high speed computers to more than 100,000 decimal places.

10. (a) Show that

$$\log_e 2 = -7 \log_e \frac{9}{10} + 2 \log_e \frac{24}{25} + 3 \log_e \frac{81}{80} .$$

(b) How many terms of the Taylor approximation to $\log_e(1 + x)$ do you need to use (a) to calculate $\log_e 2$, correct to 5 decimal places?

11. Find the Taylor approximants to

$$x \to \log_e \frac{1 + x}{1 - x}$$

with remainder estimate for $|x| < 1$.

Appendix 3

MATHEMATICAL INDUCTION

A3-1. The Principle of Mathematical Induction

The ability to form general hypotheses in the light of a limited number of facts is one of the most important signs of creativeness in a mathematician. Equally important is the ability to prove these guesses. The best way to show how to guess at a general principle from limited observations is to give examples.

Example A3-1a. Consider the sums of consecutive odd integers:

$$1 = 1$$

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

$$1 + 3 + 5 + 7 + 9 = 25.$$

Notice that in each case the sum is the square of the number of terms.

Conjecture: The sum of the first n odd positive integers is n^2. (This is true. Can you show it?)

Example A3-1b. Consider the following inequalities:

$$1 < 100, \quad 2 < 100, \quad 3 < 100, \quad 4 < 100, \quad 5 < 100, \quad \text{etc.}$$

Conjecture: All positive integers are less than 100. (False, of course.)

Example A3-1c. Consider the number of complex zeros, including the repetitions, for polynomials of various degrees.

Zero degree: a_0.	no zeros $(a_0 \neq 0)$.
First degree: $a_1x + a_0$.	one zero at $x = \frac{-a_0}{a_1}$.
Second degree: $a_2x^2 + a_1x + a_0$.	two zeros at $x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$.

Conjecture: Every polynomial of degree n has exactly n complex zeros when repetitions are counted. (True.)

Example A3-1d. Observe the operations necessary to compute the roots from the coefficients in Example A3-1c.

Conjecture: The zeros of a polynomial of degree n can be given in terms of the coefficients by a formula which involves only addition, subtraction, multiplication, division, and the extraction of roots. (False.)

Example A3-1e. Take any even number except 2 and try to express it as the sum of as few primes as possible:

$$4 = 2 + 2, \quad 6 = 3 + 3, \quad 8 = 3 + 5, \quad 10 = 5 + 5,$$
$$12 = 5 + 7, \quad 14 = 7 + 7, \quad \text{etc.}$$

Conjecture: Every even number but 2 can be expressed as the sum of two primes. (As yet, no one has been able to prove or disprove this conjecture.)

Common to all these examples is the fact that we are trying to assert something about all the members of a sequence of things: the sequence of odd integers, the sequence of positive integers, the sequence of degrees of polynomials, the sequence of even numbers greater than 2. The sequential character of the problems naturally leads to the idea of sequential proof. If we know something is true for the first few members of the sequence, can we use that result to prove its truth for the next member of the sequence? Having done that, can we now carry the proof on to one more member? Can we repeat the process indefinitely?

Let us try the idea of sequential proof on Example A3-1a. Suppose we know that for the first k odd integers $1, 3, 5, \ldots, 2k - 1$,

$$(1) \qquad 1 + 3 + 5 + \ldots + (2k - 1) = k^2.$$

can we prove that upon adding the next higher odd number $(2k + 1)$ we obtain the next higher square? From (1) we have at once by adding $2k + 1$ on both sides,

$$[1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2.$$

It is clear that if the conjecture of Example A3-1a is true at any stage then it is true at the next stage. Since it is true for the first stage, it must be true for the second stage, therefore true for the third stage, hence the fourth, the fifth, and so on forever.

Example A3-1f. In many good toy shops there is a puzzle which consists of three pegs and a set of graduated discs as depicted in Figure A3-1a. The problem posed is to transfer the pile of discs from one peg to another under the following rules:

1. Only one disc at a time may be transferred from one peg to another.
2. No disc may ever be placed over a smaller disc.

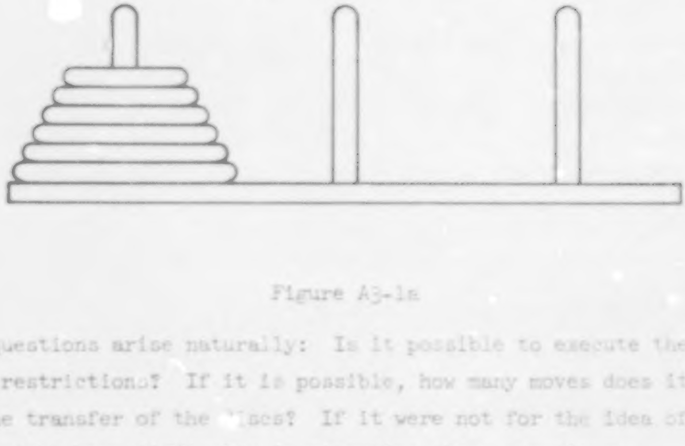


Figure A3-1a

Two questions arise naturally: Is it possible to execute the task under the stated restrictions? If it is possible, how many moves does it take to complete the transfer of the discs? If it were not for the idea of sequential proof, one might have difficulty in attacking these questions.

As it is, we observe that there is no problem in transferring one disc.

If we have to transfer two discs, we transfer one, leaving a peg free for the second disc; we then transfer the second disc and cover with the first.

If we have to transfer three discs, we transfer the top two, as above. This leaves a peg for the third disc to which it is then moved, and the first two discs are then transferred to cover the third disc.

The pattern has now emerged. If we know how to transfer k discs, we can transfer $k + 1$ in the following way. First, we transfer k discs leaving the $(k + 1)$-th disc free to move to a new peg: we move the $(k + 1)$-th disc and then transfer the k discs again to cover it. We see then that it is possible to move any number of graduated discs from one peg to another without violating the rules (1) and (2), since knowing how to move one disc, we have a rule which tells us how to transfer two, and then how to transfer three, and so on.

To determine the smallest number of moves it takes to transfer a pile of discs, we observe that no disc can be moved unless all the discs above it have been transferred, leaving a free peg to which to move it. Let us designate by m_k the minimum number of moves needed to transfer k discs. To move the $(k + 1)$-th disc, we first need m_k moves to transfer the discs above it to another peg. After that we can transfer the $(k + 1)$-th disc to the free peg. To move the $(k + 2)$-th disc (or to conclude the game if the $(k + 1)$-th disc is last) we must now cover the $(k + 1)$-th disc with the preceding k discs; this transfer of the k discs cannot be accomplished in less than m_k moves. We see then that the minimum number of moves for $k + 1$ discs is

$$m_{k+1} = 2m_k + 1.$$

This is a recursive expression for the minimum number of moves, that is, if the minimum is known for a certain number of discs, we can calculate the minimum for one more disc. In this way, we have defined the minimum number of sequential moves: by adding one disc we increase the necessary number of moves to one more than twice the preceding number. It takes one move to move one disc, therefore it takes three moves to move two discs, and so on.

Let us make a little table (Table A3-1e).

Table A3-1a

k	1	2	3	4	5	6	7
m_k	1	3	7	15	31	63	127

k = number of discs

m_k = minimum number of moves

Upon adding a disc we roughly double the number of moves. This leads us to compare the number of moves with the powers of two: 1, 2, 4, 8, 16, 32, 64, 128, ...; and we guess that $m_k = 2^k - 1$. If this is true for some value k, we can easily see that it must be true for the next, for we have

$$\begin{aligned} m_{k+1} &= 2m_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1, \end{aligned}$$

and this is the value of $2^n - 1$, for $n = k + 1$. We know that the formula for m_k is valid when $k = 1$, but now we can prove in sequence that it is true for 2, 3, 4, and so on.

According to persistent rumor, there is a puzzle of this kind in a most holy monastery hidden deep in the Himalayas. The puzzle consists of 64 discs of pure beaten gold and the pegs are diamond needles. The story relates that the game of transferring the discs has been played night and day by the monks since the beginning of the world, and has yet to be concluded. It also has been said that when the 64 discs are completely transferred, the world will come to an end. The physicists say the earth is about four billion years old, give or take a billion or two. Assuming that the monks move one disc every second and play in the minimum number of moves, is there any cause for panic? (Cf. Ball, W. W., Mathematical Recreations. New York: Macmillan Co., 1947; p. 303 ff.)

The principle of sequential proof, stated explicitly, is this (First Principle of Mathematical Induction): Let $A_1, A_2, A_3, \ldots,$ be a sequence of assertions, and let H be the hypothesis that all of these are true. The hypothesis H will be accepted as proved if

1. There is a general proof to show that if any assertion A_k is true, then the next assertion A_{k+1} is true;

2. There is a special proof to show that A_1 is true.

If there are only a finite number of assertions in the sequence, say ten, then we need only carry out the chain of ten proofs explicitly to have a complete proof. If the assertions continue in sequence endlessly, as in Example 1, then we cannot possibly verify directly every link in the chain of proof. It is just for this reason--in effect that we can handle an infinite chain of proof without specifically examining every link--that the concept of sequential proof becomes so valuable. It is, in fact, at the heart of the logical development of mathematics.

Through an unfortunate association of concepts this method of sequential proof has been named "mathematical induction." Induction, in its common English sense, is the guessing of general propositions from a number of observed facts. This is the way one arrives at assertions to prove. "Mathematical induction" is actually a method of deduction or proof and not a procedure of guessing, although to use it we ordinarily must have some guess to test. This usage has been in the language for a long time, and we would gain nothing by changing it now. Let us keep it then, and remember that mathematical usage is special and often does not resemble in any respect the usage of common English.

In Example A3-1a, above, the assertion A_n is

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

We proved first, that if A_k is true (that is, if the sum of the first k odd numbers is k^2), then A_{k+1} is true, so that the sum of the first $k + 1$ odd numbers is $(k + 1)^2$. Second, we observed that A_1 is true: $1 = 1^2$. These two steps complete the proof.

Mathematical induction is a method of proving a hypothesis about a list or sequence of assertions. Unfortunately it doesn't tell us how to make the hypothesis in the first place. In the example just considered, it was easy to guess from a few specific instances that the sum of the first n odd numbers is n^2, but the next problem (Example A3-1g) may not be so obvious.

Example A3-1g. Consider the sum of the squares of the first n positive integers,

$$1^2 + 2^2 + 3^2 + \ldots + n^2.$$

We find that when $n = 1$, the sum is 1; when $n = 2$, the sum is 5; when

$n = 3$, the sum is 14; and so on. Let us make a table of the first few values (Table A3-1b).

Table A3-1b

n	1	2	3	4	5	6	7	8
sum	1	5	14	30	55	91	140	204

Though some mathematicians might be immediately able to see a formula that will give us the sum, most of us would have to admit that the situation is obscure. We must look around for some trick to help us discover the pattern which is surely there; what we do will therefore be a personal, individual matter. It is a mistake to think that only one approach is possible.

Sometimes experience is a useful guide. Do we know the solutions to any similar problems? Well, we have here the sum of a sequence, and Example A3a also dealt with the sum of a sequence: the sum of the first n odd numbers is n^2. Consider the sum of the first n integers themselves (not their squares)--what is

$$1 + 2 + 3 + \ldots + n?$$

This seems to be a related problem, and we can solve it with ease. The terms form an arithmetic progression in which the first term is 1 and the common difference is also 1; the sum, by the usual formula, is therefore

$$\frac{n}{2}(n + 1) = \frac{1}{2}n^2 + \frac{1}{2}n.$$

So we have

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}n^2 + \frac{1}{2}n.$$

Is there any pattern here which might help with our present problem?

These two formulas have one common feature: both are quadratic polynomials in n. Might not the formula we want here also be a polynomial? It seems unlikely that a quadratic polynomial could do the job in this more complicated problem, but how about one of higher degree? Let's try a cubic: assume that there is a formula,

$$1^2 + 2^2 + \ldots + n^2 = an^3 + bn^2 + cn + d,$$

where a, b, c, and d are numbers yet to be determined. Substituting $n = 1, 2, 3$, and 4 successively in this formula, we get

$$1^2 = a + b + c + d$$

$$1^2 + 2^2 = 8a + 4b + 2c + d$$

$$1^2 + 2^2 + 3^2 = 27a + 9b + 3c + d$$

$$1^2 + 2^2 + 3^2 + 4^2 = 64a + 16b + 4c + d.$$

Solving, we find

$$a = \frac{1}{3}, \quad b = \frac{1}{2}, \quad c = \frac{1}{6}, \quad d = 0.$$

We therefore conjecture that

$$\begin{aligned} 1^2 + 2^2 + \ldots + n^2 &= \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \\ &= \frac{1}{6} n(n + 1)(2n + 1). \end{aligned}$$

This then is our assertion A_n; now let us prove it.

We have A_k:

$$1^2 + 2^2 + \ldots + k^2 = \frac{1}{6} k(k + 1)(2k + 1).$$

Add $(k + 1)^2$ to both sides, factor, and simplify:

$$\begin{aligned} 1^2 + 2^2 + \ldots + k^2 + (k + 1)^2 &= \frac{1}{6} k(k + 1)(2k + 1) + (k + 1)^2 \\ &= (k + 1)[\frac{1}{6} k(2k + 1) + (k + 1)] \\ &= \frac{1}{6} (k + 1)(k + 2)(2k + 3), \end{aligned}$$

and this last equation is just A_{k+1}, which is therefore true if A_k is true. Moreover, A_1, which states

$$1^2 = \frac{1}{6} (1)(2)(3),$$

is true; and A_n is therefore true for each positive integer n.

There is another formulation of the principle of mathematical induction which is extremely useful. This form involves the assumption in the sequential step that every assertion up to a certain point is true, rather than just the one assertion immediately preceding. Specifically, we have the following (Second Principle of Mathematical Induction): Again let $A_1, A_2, A_3, \ldots,$ be a sequence of assertions, and let H be the hypothesis that all of these are true. The hypothesis H will be accepted as proved if

1. There is a general proof to show that if every preceding assertion $A_1, A_2, \ldots, A_k,$ is true, then the next assertion A_{k+1} is true.
2. There is a special proof to show that A_1 is true.

It is not hard to show that either one of the two principles of mathematical induction can be derived from the other. The demonstration of this is left as an exercise.

The value of this second principle of mathematical induction is that it perm[illegible]tment of many problems which would be quite difficult to handle directly on the basis of the first principle. Such problems usually present a more complicated appearance than the kind which yield directly to an attack by the first principle.

Example A3-1h. Every nonempty set S of natural numbers (whether finite or infinite) contains a least element.

Proof. The induction is based on the fact that S contains some natural number. The assertion A_k is that if k is in S, then S contains a least element.

Initial Step: The assertion A_1 is that if S contains 1, then it contains a least number. This is certainly true, since 1 is the smallest natural number and so is smaller than any other member of S.

Sequential Step: We assume A_n is true for all natural numbers up to and including k. Now let S be a set containing k + 1. There are two possibilities:

1. S contains a natural number p less than k + 1.. In that case p is less than or equal to k. It follows that S contains a least element.
2. S contains no natural number less than k + 1. In that case k + 1 is least.

This example is valuable because it is a third principle of mathematical induction equivalent to the other two, although not an obvious one to be sure. An amusing example of a "proof" by this principle is given by Beckenbach in the American Mathematical Monthly, Vol. 52; 1945.

THEOREM. Every natural number is interesting.

Argument. Consider the set S of all uninteresting natural numbers. This set contains a least element. What an interesting number, the smallest in the set of uninteresting numbers! So S contains an interesting number after all. (Contradiction.)

The trouble with this "proof" of course is that we have no definition of "interesting"; one man's interest is another man's boredom.

One of the most important uses of mathematical induction is in definition by recursion, that is, in defining a sequence of things as follows: a definition is given for the initial object of the sequence, and a rule is supplied so that if any term is known the rule provides a definition for the succeeding one.

For example, we could have defined a^n $(a \neq 0)$ recursively in the following way:

Initial Step: $a^0 = 1$

Sequential Step: $a^{k+1} = a \cdot a^k$ $(k = 0, 1, 2, 3, \ldots)$

Here is another useful definition by recursion: Let $n!$ denote the product of the first n positive integers. We can define $n!$ recursively as follows:

Initial Step: $1! = 1$

Sequential Step: $(k + 1)! = (k + 1)(k!)$ $(k = 1, 2, 3, \ldots)$

Such definitions are convenient in proofs by mathematical induction. Here is an example which involves the two definitions we have just given.

Example A3-1i. For all positive integral values n, $2^{n-1} \leq n!$ The proof by mathematical induction is direct. We have the following steps.

Initial Step: $2^0 = 1 \leq 1! = 1$

Sequential Step: Assuming that the assertion is true at the k-th step, we seek to prove it for the $(k + 1)$-th step. By definition, we have

$$(k + 1)! = (k + 1)(k!).$$

From the hypothesis, $k! \geq 2^{k-1}$, and consequently,

$$(k + 1)! = (k + 1)(k!) \geq (k + 1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k$$

since $k \geq 1$ (k is a positive integer). We conclude that $(k + 1)! \geq 2^k$. The proof is complete.

Before we conclude these remarks on mathematical induction, a word of caution. For a complete proof by mathematical induction it is important to show the truth of both the initial step and the sequential step of the induction principle being used. There are many examples of mathematical induction gone haywire because one of these steps fails. Here are two examples.

Example A3-1j.

Assertion: All natural numbers are even.

Argument: For the proof we utilize the second principle of mathematical induction and take for A_k the assertion that all natural numbers less than or equal to k are even. Now consider the natural number $k + 1$. Let i be any natural number with $i \leq k$. The number j such that $i + j = k + 1$ can easily be shown to be a natural number with $j \leq k$. But if $i \leq k$ and $j \leq k$, both i and j are even; and hence $k + 1 = i + j$, the sum of two even numbers, and must itself be even!

Find the hole in this argument.

Example A3-1k.

Assertion: All girls are the same.*

Argument: Given girls designated by a and b,, let $a = b$ mean that a and b are the same. Consider any set S_1 containing just one girl. Clearly, if a and b denote girls in S_1, then $a = b$. Now suppose it is true for any set of k girls that they are all the same. Let S_{k+1} be a set containing $k + 1$ girls $g_1, g_2, \ldots, g_k, g_{k+1}$. By hypothesis the k girls, $g_1, g_2, \ldots, g_k$, are all the same, but by the same argument so are the k girls $g_2, g_3, \ldots, g_k, g_{k+1}$. It follows that $g_1 = g_2 = \ldots = g_k = g_{k+1}$. We conclude that all girls of a set containing any positive integral number of them are the same. Since there is only a positive integral number of girls in the whole world, the assertion is proved.

Find the flaw in this argument.

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*We are not trying to express an overly blasé attitude about girls. The original of this example (attributed to the famous logician Tarski) had it that all positive integers are the same; however, isn't it more interesting to write about girls?

Exercises A3-1

1. Prove by mathematical induction that $1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n + 1)$.

2. By mathematical induction prove the familiar result, giving the sum of an arithmetic progression to n terms:

$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) = \frac{n}{2} [2a + (n - 1)d].$$

3. By mathematical induction prove the familiar result, giving the sum of a geometric progression to n terms:

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

Prove the following four statements by mathematical induction.

4. $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3} (4n^3 - n)$

5. $2n \leq 2^n$

6. If $p > -1$, then, for every positive integer n, $(1 + p)^n \geq 1 + np$.

7. $1 + 2 \cdot 2 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = 1 + (n - 1)2^n$

Prove the following by the second principle of mathematical induction.

8. For all natural numbers n, the number $n + 1$ either is a prime or can be factored into primes.

9. For each natural number n greater than one, let U_n be a real number with the property that for at least one pair of natural numbers p, q with $p + q = n$, $U_n = U_p + U_q$.
When $n = 1$, we define $U_1 = a$ where a is some given real number.
Prove that $U_n = na$ for all n.

10. Attempt to prove 8 and 9 from the first principle to see what difficulties arise.

In the next three problems, first discover a formula for the sum, and then prove by mathematical induction that you are correct.

11. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)}$

12. $1^3 + 2^3 + 3^3 + \ldots + n^3$. (Hint: Compare the sums you get here with **Examples** A3-1a and A3-1g in the text, or, alternatively, assume that the required result is a polynomial of degree 4.)

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1)$. (Hint: Compare this with **Example** A3-1g in the text.)

14. Prove for all positive integers n,

$$(1 + \frac{3}{1})(1 + \frac{5}{4})(1 + \frac{7}{9}) \ldots (1 + \frac{2n + 1}{n^2}) = (n + 1)^2.$$

15. Prove that $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2^n}) = \dfrac{1 - x^{2^n + 1}}{1 - x}$.

16. Prove that $n(n^2 + 5)$ is divisible by 6 for all integral n.

17. Any infinite straight line separates the plane into two parts; two intersecting straight lines separate the plane into four parts; and three non-concurrent lines, of which no two are parallel, separate the plane into seven parts. Determine the number of parts into which the plane is separated by n straight lines of which no three meet in a single common point and no two are parallel; then prove your result. Can you obtain a more general result when parallelism is permitted? If concurrence is permitted? If both are permitted?

18. Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{p_n}{q_n}, \ldots$$

where each fraction is obtained from the preceding by the rule

$$p_n = p_{n-1} + 2q_{n-1}$$
$$q_n = p_{n-1} + q_{n-1}.$$

Show that for n sufficiently large, the difference between $\frac{p_n}{q_n}$ and $\sqrt{2}$ can be made as small as desired. Show also that the approximation to $\sqrt{2}$ is improved at each successive stage of the sequence and that the error alternates in sign. Prove also that p_n and q_n are relatively prime, that is, the fraction $\frac{p_n}{q_n}$ is in lowest terms.

19. Let p be any polynomial of degree m. Let $q(n)$ denote the sum

(1) $$q(n) = p(1) + p(2) + p(3) + \dots + p(n).$$

Prove that there is a polynomial q of degree $m + 1$ satisfying (1).

20. Let the function $f(n)$ be defined recursively as follows:

Initial Step: $f(1) = 3$

Sequential Step: $f(n + 1) = 3^{f(n)}$

In particular, we have $f(3) = 3^{3^3} = 3^{27}$, etc.

Similarly, $g(n)$ is defined by

Initial Step: $g(1) = 9$

Sequential Step: $g(n + 1) = 9^{g(n)}$

Find the minimum value m for each n such that $f(m) \geq g(n)$.

21. Prove for all natural numbers n, that $\dfrac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$ is an integer. (Hint: Try to express $x^n - y^n$ in terms of $x^{n-1} - y^{n-1}$, $x^{n-2} - y^{n-2}$, etc.)

A3-2. Sums and Sum Notation

(1) Sum Notation

In the preceding section we made frequent use of extended sums in which the terms exhibit a repetitive structure. For example, consider the sum

$$1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + \ldots + n(2n - 1). \qquad (1)$$

We adopt a concise notation which indicates the repetition instead of spelling it out. In this notation the sum (1) is written

$$\sum_{k=1}^{n} k(2k - 1).$$

This symbol means, "the sum of all terms of the form $k(2k - 1)$ where k takes on the integer values from 1 to n inclusive." The Greek capital "Σ" (sigma) corresponds to the Roman "S" and is intended to suggest the word "sum."

The notation can be used more generally to express the sum of any quantities ϕ_k where k takes on consecutive integral values; we may begin with any integer m and end with any integer n where $n \geq m$. Thus

$$\sum_{k=m}^{n} \phi_k = \phi_m + \phi_{m+1} + \phi_{m+2} + \cdots + \phi_n.$$

(Note the trivial special case, $n = m$, a "sum" of one term: $\sum_{k=m}^{n} \phi_k = \phi_m$.)

Example A3-2a. If each of the regions R_k in (1) is a rectangle with height h_k and width w_k, the sum of the areas may we written

$$w_1h_1 + w_2h_2 + w_3h_3 + \ldots + w_nh_n = \sum_{k=1}^{n} w_kh_k.$$

Here are other typical examples:

$$\begin{aligned} \sum_{k=0}^{3} \frac{k}{1 + k^2} &= \frac{0}{1 + 0} + \frac{1}{1 + 1} + \frac{2}{1 + 4} + \frac{3}{1 + 9} \\ &= 0 + \frac{1}{2} + \frac{2}{5} + \frac{3}{10} \\ &= \frac{6}{5}, \end{aligned}$$

$$\sum_{j=2}^{5} (j + 3) = 5 + 6 + 7 + 8 = 26.$$

A linear combination of n functions:

$$\sum_{j=1}^{n} a_j f_j(x) = a_1 f_1(x) + a_2 f_2(x) + \dots + a_n f_n(x).$$

A polynomial of degree no greater than m:

$$\sum_{i=0}^{m} c_i x^i = c_0 + c_1 x + c_2 x^2 + \dots + c_m x^m.$$

Example A3-2b. A simple but important sum is $\sum_{j=1}^{n} c$, where c is a constant, that is, a quantity independent of the index j of summation. The quantity $\sum_{j=1}^{n} c$ is the sum of n terms each of which is c; it therefore has the value nc.

In any summation the values of the terms and the total are not affected by the choice of the index letter; thus

$$\sum_{k=m}^{n} \phi_k = \sum_{j=m}^{n} \phi_j.$$

We are free to choose the index letter and its initial value to suit our own convenience.

Example A3-2c.

(a) $$\sum_{j=0}^{2} a_j = a_0 + a_1 + a_2 = \sum_{p=1}^{3} a_{p-1} = \sum_{n=0}^{2} a_{2-n}$$

(b) $$\sum_{i=0}^{n} a_i^{\,n-i} = a_0^{\,n} + a_1^{\,n-1} + \dots + a_n^{\,0} = \sum_{j=0}^{n} (a_{n-j})^j$$

Summation is a linear process; the proof is left as the first excercise below.

Exercises A3-2a

1. Prove

$$\sum_{k=1}^{n} (\alpha f_k + \beta g_k) = \alpha \sum_{k=1}^{n} f_k + \beta \sum_{k=1}^{n} g$$

2. Write each of the following sums in expanded form and evaluate:

(a) $\sum_{k=1}^{5} 2k$

(b) $\sum_{j=5}^{10} j^2$

(c) $\sum_{r=-1}^{3} (r^2 + r - 12)$

(d) $\sum_{m=2}^{5} m(m - 1)(m - 2)$

(e) $\sum_{i=0}^{10} 2^i$

(f) $\sum_{r=0}^{4} \frac{4!}{r!(4 - r)!}$

3. Which of the following statements are true and which are false? Justify your conclusions.

(a) $\sum_{j=3}^{10} 4 = 7 \cdot 4 = 28$

(b) $\sum_{j=m}^{n} 4 = 4\big((n - m) + 1\big)$

(c) $\sum_{k=1}^{10} k^2 = 10 \sum_{k=1}^{9} k^2$

(d) $\sum_{k=1}^{1000} k^2 = 5 + \sum_{k=3}^{1000} k^2$

(e) $\sum_{k=1}^{n} k^3 = n^3 + \sum_{j=2}^{n} (j - 1)^3$

(f) $\sum_{m=1}^{10} k^2 = \left(\sum_{m=1}^{10} k\right)^2$

(g) $\sum_{m=1}^{10} k^3 = \left(\sum_{m=1}^{10} k\right)^2$

(h) $\sum_{i=0}^{n} i(i - 1)(n - i) = \sum_{i=2}^{n-1} i(i - 1)(n - i)$

(i) $\sum_{k=0}^{m} f(a_{m-k}) = \sum_{k=0}^{m} f(a_k)$

(j) $n\sum_{k=0}^{n} A_k - \sum_{k=0}^{n} k\ A_k = \sum_{k=0}^{n} k\ A_{n-k}$

(k) $\sum_{k=0}^{m} k^2(A_k - A_{m-k}) = m^2\sum_{k=0}^{m} A_{m-k} - 2m\sum_{k=0}^{m} k\ A_{m-k}$

4. Evaluate $\sum_{k=1}^{n} f(\frac{k}{n})\frac{(b - a)}{n}$ if $f(x) = x^2$, $a = 0$, $b = 1$, and

(a) $n = 2$

(b) $n = 4$

(c) $n = 8$

5. Subdivide the interval $[0,1]$ into n equal parts. In each subinterval obtain upper and lower bounds for x^2. Using sigma notation use these upper and lower bounds to obtain expressions for upper and lower estimates of the area under the curve $g = x^2$ on $[0,1]$. If you can evaluate these sums without reading elsewhere, do so.

6. (a) Write out the sum of the first 7 terms of an arithmetic progression with first term a and common difference d. Express the same sum in sigma notation.

(b) In sigma notation, write the expression for the sum of the first n terms of a geometric progression with first term a and common ratio r.

7. (a) Consider a function f defined by

$$f(n) = \sum_{r=1}^{n}\left\{(r - 1)(r - 2)(r - 3)(r - 4)(r - 5) + r\right\}.$$

Find $f(n)$ for $n = 1, 2, \ldots, 5$.

(b) Give an example of a function g (similar to that in (a)) such that

$g(n) = 1 \quad n = 1, 2, \ldots, 10^6,$

$g(10^6 + 1) = 0.$

8. Write each of the following sums in expanded form and evaluate.

(a) $\sum_{n=1}^{4}\left\{\sum_{r=1}^{3} r(n - r)\right\}$

(b) $\sum_{n=1}^{N}\left\{\sum_{r=1}^{R} (rn - 1)\right\}$

9. The double sum $\sum_{i=0}^{m}\sum_{j=0}^{n} F(i,j)$ is a shorthand notation for

$$\sum_{i=0}^{m}\left\{F(i,0) + F(i,1) + \dots + F(i,n)\right\}$$

or

$$\begin{array}{l} F(0,0) + F(0,1) + \dots + F(0,n) \\ + F(1,0) + F(1,1) + \dots + F(1,n) \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ + F(m,0) + F(m,1) + \dots + F(m,n). \end{array}$$

In particular $\sum_{i=1}^{2}\sum_{j=1}^{3} i \cdot j = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 = 18$. Evaluate:

(a) $\sum_{i=1}^{m}\sum_{j=1}^{n} i \cdot j$

(b) $\sum_{i=1}^{m}\sum_{j=1}^{n} (i + j)$

(c) $\sum_{i=1}^{m}\sum_{j=1}^{n} \max(i,j)$

(d) $\sum_{i=1}^{m}\sum_{j=1}^{n} \min(i,j)$

10. (a) Show that $\frac{1}{k(k - 1)} = \frac{1}{k - 1} - \frac{1}{k}$, $k \neq 0, 1$.

(b) Evaluate $\sum_{k=2}^{1000} \frac{1}{k(k - 1)}$.

11. If $S(n) = \sum_{i=1}^{n} f(i)$, determine $f(m)$ in terms of the sum function S.

12. Determine $f(m)$ in the following summation formulae:

(a) $1 = \sum_{i=1}^{n} f(i)$

(b) $n = \sum_{i=1}^{n} f(i)$

(c) $n^2 = \sum_{i=1}^{n} f(i)$

(d) $an^2 + bn + c = \sum_{i=1}^{n} f(i)$

(e) $\cos nx = \sum_{i=1}^{n} f(i)$

(f) $\sin (an + b) = \sum_{i=1}^{n} f(i)$

(g) $n! = \sum_{i=1}^{n} f(i)$

13. Binomial Theorem: We define $\binom{n}{r} = \frac{n!}{(n - r)!r!}$ where r, n are integers such that $0 \leq r \leq n$. Also $0! = 1$ and $\binom{n}{r} = 0$ if $r > n$. Show that

(a) $\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{1} = \binom{n}{n-1} = n$

(b) $\binom{n}{r} = \binom{n}{n-r}$

$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

(c) Establish the Binomial Theorem

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + nx^{n-1} y + \dots + nxy^{n-1} + y^n,$$

$n = 0, 1, 2, \dots,$ by mathematical induction.

14. Using the Binomial Theorem, give the expansions for the following:

(a) $(x + y)^3$

(b) $(x - y)^3$

(c) $(2x - 3y)^3$

(d) $(x - 2y)^5$

15. Evaluate the following sums.

(a) $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$

(b) $\binom{n}{0} - \binom{n}{1} + \dots + (-1)^n \binom{n}{n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$

16. Sum $\sum_{r=0}^{n} r\binom{n}{r}$ by first showing $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$ and using 15(a).

17. If $P_n(x)$ denotes a polynomial of degree n such that $P_n(x) = 2^x$ for $x = 0, 1, 2, \ldots, n$ find $P_n(n + 1)$.

(ii) Summation

Exercises A3-1, No. 10 illustrates a particularly useful summation technique, i.e., representation as a telescoping sum. It was possible to write

$$\sum_{k=2}^{1000} \frac{1}{k(k-1)} = \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \ldots + \frac{1}{1000 \cdot 999}$$

in the form

$$\sum_{k=2}^{1000} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{998} - \frac{1}{999}\right) + \left(\frac{1}{999} - \frac{1}{1000}\right).$$

Each quantity subtracted in one parenthesis is added back in the next, so that the first two terms telescope from a sum of four numbers to a sum of two numbers, the first three terms telescope from a sum of six numbers to a sum of two numbers, etc. Finally, the entire summation telescopes (or collapses) into a sum of two numbers--the first number in the first term and the second number in the last term. Symbolically, a telescoping sum has the form

(1) $$\sum_{k=m}^{n} \Big\{f(k) - f(k - 1)\Big\} = f(n) - f(m - 1).$$

In the above example, we have $m = 2$, $n = 1000$, and $f(k) = -\frac{1}{k}$ so that the sum telescopes to $f(1000) - f(1) = -\frac{1}{1000} + 1 = \frac{999}{1000}$.

We now use (1) to establish a short dictionary of summation formulae by considering different functions $f(k)$. Also, we let $m = 1$ without loss of generality. Let $f(k) = k$, then

(2) $$\sum_{k=1}^{n} \Big\{k - (k - 1)\Big\} = \sum_{k=1}^{n} 1 = n.$$

This result is nothing new. Now let $f(k) = k^2$, then

$$\sum_{k=1}^{n}\left\{k^2 - (k - 1)^2\right\} = \sum_{k=1}^{n}(2k - 1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = n^2$$

or, equivalently,

$$\sum_{k=1}^{n} k = \frac{1}{2}\, n(n + 1). \tag{3}$$

By linearly combining (2) and (3), we obtain the sum of a general arithmetic progression

$$\sum_{k=1}^{n}(ak + b) = a\left\{\frac{n(n + 1)}{2}\right\} + bn.$$

To obtain the sum $\sum_{k=1}^{n} k^2$, we let $f(k) = k^3$. Then,

$$\sum_{k=1}^{n}\left\{k^3 - (k - 1)^3\right\} = \sum_{k=1}^{n}(3k^2 + 3k + 1) = n^3,$$

$$3\sum_{k=1}^{n}k^2 - 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 = n^3.$$

Using (2) and (3), we obtain

$$\sum_{k=1}^{n}k^2 = \frac{1}{3}\left\{n^3 + \frac{3n(n + 1)}{2} - n\right\} = \frac{n(n + 1)(2n + 1)}{6}.$$

We now can establish a sequential method of obtaining sums of the form $\sum_{k=1}^{n} P(k)$ whose terms are values $P(k)$ of a polynomial function. Because a polynomial is a linear combination of powers, and summation is a linear process, it is sufficient to give a sequential method for $\sum_{k=1}^{n} k^r$, r a nonnegative integer.

Choosing $f(k) = k^{r+1}$ in summation, formula (1) gives us

$$\sum_{k=1}^{n}\left\{k^{r+1} - (k - 1)^{r+1}\right\} = n^{r+1}.$$

Using the Binomial Theorem, we obtain

$$k^{r+1} - (k - 1)^{r+1} = (r + 1)k^r + P(k) \tag{4}$$

where $P(k)$ is a polynomial of degree $r - 1$. Thus, the sum $\sum_{k=1}^{n} k^r$ can be expressed in terms of sums of lower degree. Since we already have the sum for $r = 0$, 1, and 2, we can repeat the method sequentially to obtain the sum for any r (compare with Exercises A3-1, No. 19).

We can enlarge our summation table by choosing other functional forms $f(k)$, e.g., $\sin(ak + b)$. By (1),

$$\sum_{k=1}^{n} \left\{\sin(ak + b) - \sin\Big(a(k - 1) + b\Big)\right\} = \sin(an + b) - \sin b.. \tag{5}$$

Using the identity

$$\sin A - \sin B = 2 \sin \frac{A - B}{2} \cos \frac{A + B}{2} ,$$

in Equation (5), we obtain

$$\sum_{k=1}^{n} \cos(ak + b - \tfrac{a}{2}) = \cos(b + \tfrac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}} . \tag{6}$$

If $b = \frac{a}{2}$, (6) reduces to

$$\sum_{k=1}^{n} \cos ak = \cos\Big((a + 1) \tfrac{n}{2}\Big) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}} . \tag{7}$$

If $b = \frac{a}{2} + \frac{\pi}{2}$, (6) reduces to

$$\sum_{k=1}^{n} \sin ak = \sin(b + \tfrac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}} . \tag{8}$$

By choosing other functions $f(k)$, we can enlarge our list of summation formulae. We leave this for exercises.

Exercises A3-2b

1. Write the following sums in telescoping form, i.e., in the form $\sum_{k=1}^{n} \left\{u(k) - u(k-1)\right\}$, and evaluate.

   (a) $\sum_{k=1}^{n} k(k+1)$

   (b) $\sum_{k=1}^{n} k(2k-1)$

   (c) $\sum_{k=1}^{n} 2k(2k+1)$

   (d) $\sum_{k=1}^{n} k(k+1)(k+2)$

   (e) $\sum_{k=1}^{n} k^3$

   (f) $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$

   (g) $\sum_{k=1}^{n} k \cdot k!$

   (h) $\sum_{k=1}^{n} r^k$

2. Using $\sum_{k=1}^{n} \left\{u(k) - u(k-1)\right\} = u(n) - u(0)$, establish a short dictionary of summation formulae by considering the following functions u:

   (a) $(a + kd)\Big(a + (k+1)d\Big) \ldots \Big(a + (k+p)d\Big)$

   (b) The reciprocal of (a).

   (c) r^k

   (d) kr^k

   (e) $k^2 r^k$

   (f) $k!$

   (g) $(k!)^2$

   (h) $\arctan k$

   (i) $k \sin k$

3. Simplify:

$$\frac{\sin x + \sin 3x + \ldots + \sin\big((2n-1)x\big)}{\cos x + \cos 3x + \ldots + \cos\big((2n-1)x\big)} .$$

4. Another method for summing $\Sigma\, P(k)$ (P , a polynomial) can be obtained by using a special case of problem 2a, i.e.,

$$\sum_{k=1}^{n} \left\{(k + 1)(k)(k - 1) \ldots (k - r + 1) - (k)(k - 1)(k - 2) \ldots (k - r)\right\}$$

$$= (n + 1)(n)(n - 1) \ldots (n - r + 1),$$

or $$\sum_{k=1}^{n} k(k - 1) \ldots (k - r + 1) = \frac{(n + 1)(n)(n - 1) \ldots (n - r + 1)}{r + 1} .$$

First, we show how to represent any polynomial $P(k)$ of r^{th} degree in the form

$$\text{(i)} \quad P(k) = a_0 + a_1 k + \frac{a_2 k(k - 1)}{2!} + \ldots + \frac{a_r k(k - 1) \ldots (k - r + 1)}{r!} .$$

If $k = 0$, then $a_0 = P(0)$; if $k = 1$, then $a_1 = P(1) - P(0)$; if $k = 2$, than $a_2 = P(2) - 2P(1) + P(0)$. In general, it can be shown that

$$\text{(ii)} \quad a_m = P(m) - \binom{m}{1}P(m - 1) + \binom{m}{2}P(m - 2) - \ldots + (-1)^m P(0),$$

$$m = 0, 1, \ldots, r.$$

Since both sides of (i) are polynomials of degree r and (i) is satisfied for $m = 0, 1, \ldots, r$, it must be an identity.

Now sum $\displaystyle\sum_{k=1}^{n} P(k)$.

5. Using Prob. 4, find the following sums:

(a) $\displaystyle\sum_{k=1}^{n} k^2$

(b) $\displaystyle\sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2$

(c) $\displaystyle\sum_{k=1}^{n} k^4$

6. (a) Establish Equation (ii) of Number 4.

(b) Show that a_m is zero for $m > r$.

Appendix 4

FURTHER TECHNIQUES OF INTEGRATION

A4-1. Substitutions of Circular Functions

Although it is not always possible to integrate a given function in terms of elementary functions, there are important broad classes of explicitly integrable functions. All powers and hence, clearly, all polynomials are explicitly integrable. It is not so clear but it is true that all rational functions are explicitly integrable (see Section A4-3). It follows that all integrals which can be transformed by substitution into integrals of rational functions are explicitly integrable. In this section we shall show that an integral of any rational combination of x and $\sqrt{Q(x)}$, where

$$Q(x) = Ax^2 + Bx + C,$$

can be transformed into an integral of a rational combination of circular functions, and further that an integral of a rational combination of circular functions can be transformed into an integral of a rational function.

We should consider the substitution of a circular function whenever an integrand is a combination of x and one of the expressions $\sqrt{a^2 - x^2}$ $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, $(a > 0)$ suggestive of the Pythagorean expression for one of the sides of a right triangle in terms of the other two.

Example A4-1a. Consider

$$I = \int_0^{a/2} \frac{dx}{\sqrt{a^2 - x^2}}.$$

We utilize the substitution

$$x = a \sin \theta, \quad \sqrt{a^2 - x^2} = a \cos \theta \qquad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right),$$

$$dx = a \cos \theta \, d\theta;$$

(See Figure A4-1a.) Observing that for $x = \frac{a}{2}$, $\theta = \frac{\pi}{6}$, we obtain by the substitution rule,

$$I = \int_0^{\pi/6} \frac{a \cos \theta}{a \cos \theta} \, d\theta = \int_0^{\pi/6} d\theta = \frac{\pi}{6}.$$

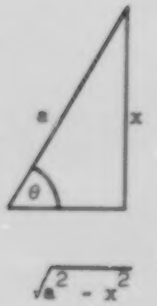


Figure A4-1a

Example A4-1b. For the integral

$$I = \int \frac{dx}{(x^2 + a^2)^{3/2}}$$

we employ the substitution (see Figure A4-1b)

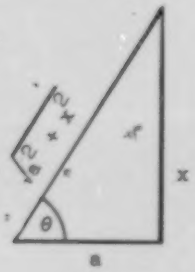


Figure A4-1b

$$x = a \tan \theta \qquad (-\frac{\pi}{2} < \theta < \frac{\pi}{2}),$$

$$dx = \frac{a}{\cos^2 \theta} d\theta$$

$$\sqrt{a^2 + x^2} = \frac{a}{\cos \theta} .$$

Thus we obtain

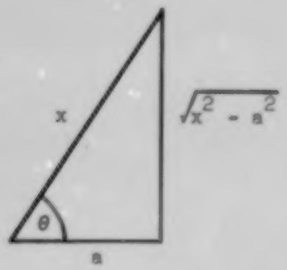
$$I = \int \frac{\cos^3 \theta}{a^3} \, \frac{a}{\cos^2 \theta} d\theta = \frac{1}{a^2} \int \cos \theta \, d\theta$$

$$= \frac{\sin \theta}{a^2} + C = \frac{x}{a^2 \sqrt{a^2 + x^2}}$$

Example A4-1c The integration

$$I = \int \frac{1}{x^2 \sqrt{x^2 - a^2}} dx$$

is performed with the aid of the substitution (see Figure A4-1c)*



$$x = \frac{a}{\cos\theta} = a\sec\theta,$$

$$dx = \frac{a\sin\theta}{\cos^2\theta}\,d\theta = a\sec\theta\tan\theta\,d\theta$$

$$\sqrt{x^2 - a^2} = a\tan\theta.$$

Figure A4-1c

We have

$$I = \int\left(\frac{\cos^2\theta}{a^2}\right)\left(\frac{1}{a\tan\theta}\right)\left(\frac{a\sin\theta}{\cos^2\theta}\right)d\theta$$

$$= \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{\sin\theta}{a^2} + C = \frac{\sqrt{x^2 - a^2}}{a^2x} .$$

Example A4-1d. Consider the integral

$$I = \int\frac{1}{\sqrt{x^2 - a^2}}\,dx .$$

Using the substitution of Example A4-1c we obtain

$$I = \int\frac{1}{a\tan\theta}\left(\frac{a\sin\theta}{\cos^2\theta}\right)d\theta = \int\frac{1}{\cos\theta}\,d\theta .$$

We can write

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{\cos^2\theta} = \frac{\cos\theta}{1 - \sin^2\theta} = \frac{\cos\theta}{2}\left[\frac{1}{1 - \sin\theta} + \frac{1}{1 + \sin\theta}\right] .$$

With this much as a hint we leave the integration as an exercise. (See also Section A4-3.)

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*Here take $0 < \theta < \frac{\pi}{2}$ for $x > 0$, and $\frac{\pi}{2} < \theta < \pi$ for $x < 0$.

THEOREM A4-1a. An integral of any rational combination of x and $\sqrt{Q(x)}$ where

(1) $$Q(x) = Ax^2 + Bx + C, \qquad (A \neq 0)$$

can be transformed by a substitution $x = f(\theta)$, where f is a circular function, into an integral of a rational combination of $\sin\theta$ and $\cos\theta$.

Proof. We are concerned with integrals of the form

(2) $$I = \int \phi\left(x, \sqrt{Q(x)}\right) dx$$

where ϕ is a rational expression and $Q(x)$ is given by (1). For the proof we first make a preliminary linear transformation to replace $Q(x)$ by one of the standard forms of Examples A4-1a, b, c.

We "complete the square" to obtain

(3) $$Q(x) = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right].$$

We set $a = \sqrt{\left|\frac{C}{A} - \frac{B^2}{4A^2}\right|}$, $b = \frac{B}{2A}$, $c = \sqrt{|A|}$, and $x = u - b$ in (3), and separate the problem into three cases.

Case (1).

If $A < 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} < 0$ we have

$$\sqrt{Q(x)} = c\sqrt{a^2 - u^2}.$$

Since $dx = du$, the substitution $x = u - b$ yields

(4) $$I = \int \phi(u - b, c\sqrt{a^2 - u^2})du.$$

Now, employing the substitution $u = a \sin\theta$ of Example A4-1a, we transform the integral into the form

(5) $$I = a\int \phi(a \sin\theta - b, c a \cos\theta) \cos\theta \, d\theta, \quad \theta = \arcsin\frac{x + b}{a}.$$

Since ϕ involves only rational operations, we have established the theorem in this case.

Case (ii).

If $A > 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} < 0$, the substitution

$$x + b = u = a \tan \theta,$$

as in Example A4-1b, confirms the theorem for this case.

Case (iii).

If $A > 0$ and $\frac{C}{A} - \frac{B^2}{4A^2} > 0$, the substitution

$$x + b = u = \frac{a}{\cos \theta},$$

as in Examples A4-1c, yields the desired result.

The integral (2) can be also transformed into an integral of a rational combination of $\sinh t$ and $\cosh t$ by an appropriate transformation $x = f(t)$ where f is a hyperbolic function. The proof is left as an exercise.

> THEOREM A4-1b. An integral of a rational combination of $\sin x$ and $\cos x$ can be transformed into an integral of a rational function by a suitable substitution.

Proof. We consider integrals of the form

$$\int \psi(\sin x , \cos x)dx \tag{8}$$

where ψ is a rational expression. We observe that $\sin x$ and $\cos x$ are rational expressions in $t = \tan \frac{x}{2}$; namely,

$$\sin x = \frac{2t}{1 + t^2} , \cos x = \frac{1 - t^2}{1 + t^2} . \tag{9}$$

Furthermore,

$$dx = d(2 \arctan t) = \frac{2}{1 + t^2} dt. \tag{10}$$

Consequently we may transform the integral (8) into the integral of a rational function by employing the substitution

$$x = 2 \arctan t; \tag{11}$$

thus, entering (9) and (10) in (8) we obtain the integral in the form

$$\int R\left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right)\frac{2}{1 + t^2}\,dt. \tag{12}$$

Theorems 10-3a and 10-3b do not necessarily point the way to the simplest method of integration for a function of one of the types considered here; they simply indicate a line of approach which is sure to work but may lead to enormous complication. Often some special device leads to the solution far more simply and directly.

Exercises A4-1

1. Integrate the following functions, the numbers a and b being positive.

   (a) $\dfrac{\sqrt{a^2 - x^2}}{x^2}$

   (b) $\dfrac{\sqrt{1 + x^2}}{x^4}$

   (c) $x^2\sqrt{a^2 - x^2}$

   (d) $\dfrac{1}{x^2\sqrt{x^2 - a^2}}$

   (e) $\dfrac{x}{(x^2 + a^2)\sqrt{x^2 - b^2}}$

   (f) $\dfrac{1}{(x^2 + a^2)\sqrt{a^2x^2 + 1}}$

   (g) $\dfrac{x + 2}{\sqrt{m^2 \pm x^2}}$

   (h) $x^3\sqrt{(4 - x^2)^5}$

   (i) $\dfrac{1}{\sqrt{a^2x - x^2}}$

   (j) $\dfrac{x^2 + ax + b}{x^2 + 1}$

   (k) $\sqrt{a^2x + x^2}$

2. Let $R(x,y)$ denote a rational function in x and y. Reduce the following integrals to integrals of rational functions.

   (a) $\displaystyle\int R(x, \sqrt{ax + b})\,dx$, $a \neq 0$.

   (b) $\displaystyle\int R\left(x, \sqrt[n]{\frac{ax + b}{cx + d}}\right)dx$, n an integer, $ad - bc \neq 0$.

3. Using the result of Number 2, integrate $\dfrac{x}{\sqrt{ax + b} + \sqrt{(ax + b)^3}}$.

4. Reduce to rational form $\displaystyle\int \frac{dx}{\sqrt{\frac{1 - x}{1 + x}} + \sqrt[4]{\frac{1 - x}{1 + x}}}$.

5. Express as elementary functions

   (a) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$

   (b) $\displaystyle\int \frac{dx}{1 + \sin x}$

   (c) $\displaystyle\int \frac{dx}{1 - \cos 2x}$

   (d) $\displaystyle\int \frac{dx}{x\sqrt[4]{1 + x^4}}$

   (e) $\displaystyle\int \frac{dx}{\sqrt[4]{1 + x^4}}$

6. (a) The integral $\displaystyle\int \frac{P(x)}{\sqrt{ax^2 + 2bx + c}}\,dx$, where $P(x)$ is a polynomial of degree n and $a \neq 0$ can be reduced to a rational trigonometric form as described in the text. It can be also reduced to the integration of $\dfrac{1}{\sqrt{ax^2 + 2bx + c}}$; namely for some polynomial Q of degree $(n - 1)$ and constant k.

$$\frac{P(x)}{\sqrt{ax^2 + 2bx + c}} = D\left(Q(x)\sqrt{ax^2 + 2bx + c}\right) + \frac{k}{\sqrt{ax^2 + 2bx + c}}.$$

   Show how to find Q and k.

   (b) Using (a), integrate $\dfrac{t^5 - t^3 + t}{\sqrt{1 - t^2}}$.

   (c) Calculate the integral of (b) by using trigonometric substitutions, and compare the merits of the two methods.

7. Integrate

(a) $\frac{1}{\sin x}$

(b) $\frac{1}{\cos x}$ (by a method other than that of Example A4-1d).

A4-2. Integration by Parts

(1) The basic formula. The method of integration by parts is used to integrate certain kinds of products. The method corresponds to the formula for the derivative of a product.

THEOREM A4-2a. If f and g have continuous derivatives over a common interval containing a and b then

$$(1) \quad \int_a^b f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x)g(x)dx.$$

The theorem follows directly from the product rule ((4) of Section 8-4) and the Fundamental Theorem of Calculus.

In Leibnizian notation, for $u = f(x)$, $du = f'(x)dx$ and $v = g(x)$, $dv = g'(x)dx$ we obtain for the definite integral corresponding to (1),

$$(2) \qquad \int u\,dv = uv - \int v\,du.$$

Integration by means of (2) is called integration by parts.

Example A4-2a. To integrate $x \to \log_e x$ observe that $\log_e x$ has an especially simple derivative and set $u = \log_e x$ and $dv = 1 \cdot dx$. For v, then, we take $v = x$. Consequently, from (2)

$$\begin{aligned}\int \log_e x\,dx &= x \log_e x - \int \frac{x}{x}\,dx \\ &= x \log_e x - x\end{aligned}$$

the formula we have already obtained.

In application, (2) is used as above for the integral of a product where the product of the integral of one factor and the derivative of the other is formally integrable.

The Leibnizian notation in (2) was introduced as a shorthand for the explicit formula. But the notation suggests that we might interpret u as a function of v, and v as the inverse function of u. This idea yields an illuminating geometrical interpretation of integration by parts. Suppose that $u = f(x)$ and $v = g(x)$ where f and g have inverses. Then we can write $u = \phi(v)$ and $v = \psi(u)$ where ϕ and ψ are inverses. (The proof is left to Exercises A4-2, No. 2). Set $u_0 = f(a)$, $u_1 = f(b)$ and $v_0 = g(a)$ $v_1 = g(b)$. We have $u_i = \phi(v_i)$ and, inversely, $v_i = \psi(u_i)$ for $i = 1, 2$. Now suppose ϕ and ψ are increasing and nonnegative. Then, from the familiar interpretation of integral as area (see Figure A4-2a) we immediately have

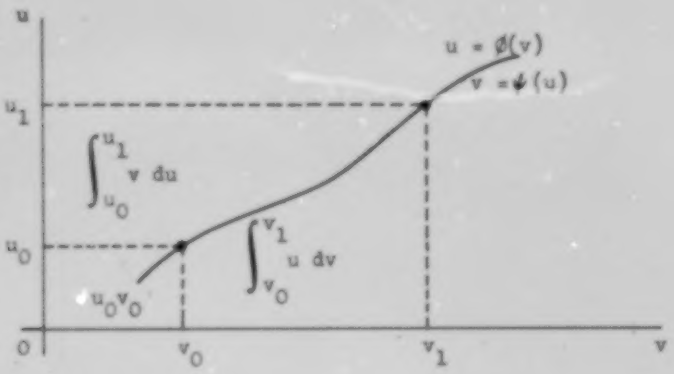


Figure A4-2a

$$u_1v_1 = \int_{v_0}^{v_1} u\,dv + \int_{u_0}^{u_1} v\,du + u_0v_0,$$ from which we at once obtain

$$\int_{v_0}^{v_1} u\,dv = [u_1v_1 - u_0u_0] - \int_{u_0}^{u_1} v\,du.$$

From the Substitution Rule we immediately recognize this equation as a form of (1). A like geometrical argument gives the same result when ϕ and ψ are decreasing.

In general, this interpretation of integration by parts gives the formal integral of any function which has a formally integrable inverse.

Example A4-2b. Consider

$$\int x^n \arcsin x \, dx, \qquad (n \text{ integral, } n \neq -1).$$

Since the arcsin has a simple algebraic derivative we set $u = \arcsin x$, $dv = x^n dx$ and take $v = \frac{x^{n+1}}{n+1}$. For the domain $0 < x \leq \frac{\pi}{2}$ we have $u = \arcsin \sqrt[n+1]{(n+1)v}$ and $v = \frac{1}{n+1} \sin^{n+1} u$. From Theorem A4-1b we know that $\int v \, du$ can be transformed into the integral of a rational function. As we shall see (Section A4-3) rational functions are always formally integrable. It follows that $\sin^{n+1} u$ is formally integrable with respect to u and hence that $x^n \arcsin x$ is formally integrable with respect to x. Reduction to the integral of a rational function is not necessarily the most efficient way to carry out these integrations, but integration by parts can be used more effectively in other ways to execute the integrations.

The idea of Example A4-2b, for $u = f(x) dv = x^n \, dx$, establishes the formal integrability of $x^n f(x)$ where f is any inverse circular function, and, in view of Example A4-2a, if $f(x) = \log x$.

Example A4-2c. Consider

$$\int x^r \log x \, dx, \qquad (r \text{ real}).$$

Since $\log x$ has a simple derivative, we set $u = \log x$, $dv = x^r dx$. If $r \neq -1$ we take $v = \frac{x^{r+1}}{r+1}$ to obtain

$$\int x^r \log x \, dx = \frac{x^{r+1}}{r+1} \log x - \frac{1}{r+1} \int x^r \, dx$$

$$= \frac{x^{r+1}}{r+1} \log x - \frac{x^{r+1}}{(r+1)^2} .$$

If $r = -1$, we may take $v = \log x$ to obtain

$$\int \frac{\log x}{x} dx = (\log x)^2 - \int \frac{\log x}{x} dx,$$

which yields

$$\int \frac{\log x}{x}\, dx = \frac{(\log x)^2}{2} + C,$$

a result which is obtained more directly from the substitution $\log x = t$.

The method of Example A4-2c, for $u = f(x)$ and $dv = x^n dx$, exhibits the formal integrability of any function of the form $x^n f(x)$, when $n \neq -1$, where $f'(x)$ is any rational combination of x and $\sqrt{Q(x)}$ and $Q(x)$ is a quadratic polynomial. Integration by parts expresses the given integral in terms of the integral of $\frac{x^{n+1}}{n+1} f'(x)$ which may be transformed into the integral of a rational function by Theorem A4-1a. From the assumed integrability of rational functions, the result follows. It follows as a slight generalization that $P(x)f(x)$ is formally integrable for any polynomial function P. From this argument we observe again that if f is a logarithmic or inverse circular, function, then $x^n f(x)$ is formally integrable. In addition, for $h(x) = \phi(x, \sqrt{Q(x)})$, a rational combination of x and $\sqrt{Q(x)}$, the expressions $x^n \log h(x)$ and $x^n \arctan h(x)$ and are all formally integrable since the derivatives of log and arctan and are rational functions.

Example A4-2d. Consider the integral

$$\int x\, e^x\, dx$$

We integrate by parts. Set $u = x$ $dv = e^x dx$ and $v = e^x$. Then by (2)

$$\int x\, e^x\, dx = xe^x - \int e^x\, dx$$
$$= xe^x - e^x$$

Integration by parts may be used to produce a simplification rather than a final complete integration as in Example A4-2c when $r = -1$.

Example A4-2e. Consider

$$I = \int e^{bx} \sin ax \, dx$$

For $u = \sin ax$, $dv = e^{bx}\,dx$, $v = \frac{e^{bx}}{b}$, we obtain

$$I = \frac{1}{b} e^{bx} \sin ax - \frac{a}{b} \int e^{bx} \cos ax \, dx$$

$$= \frac{1}{b} e^{bx} \sin ax - \frac{a}{b} J,$$

where

$$J = \int e^{bx} \cos ax \, dx$$

presents the same difficulties of formal integration as I. However, by the same technique, we can express J in terms of I and hopefully may obtain an equation which can be solved for I. Now take $u = \cos ax$ and $v = \frac{e^{bx}}{b}$ in (2) to obtain

$$J = \frac{1}{b} e^{bx} \cos ax + \frac{a}{b} \int e^{bx} \sin ax \, dx$$

$$= \frac{1}{b} e^{bx} \cos ax + \frac{a}{b} I.$$

Entering the expression for J above in the expression for I and solving for I, we obtain

$$I = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax).$$

(ii) Recurrence relations. The idea here is to express an integral of the general form $\int f_n(x)\,dx$ in terms of $\int f_{n-k}(x)\,dx$.

Example A4-2f. Consider

$$I_n = \int x^r(1 - x)^n \, dx \qquad (n \geq 0 \, , \, r \neq -1).$$

Set $u = (1 - x)^n$, $dv = x^r dx$, $v = \frac{x^{r+1}}{r + 1}$. Then

$$I_n = \frac{x^{r+1}(1 - x)^n}{r + 1} + \frac{n}{r + 1} \int x^{r+1}(1 - x)^{n-1} dx$$

where, for $n = 0$, the result yields, correctly, $I_n = \frac{x^{r+1}}{r + 1}$. Now, observe that

$$x^{r+1}(1 - x)^{n-1} = -x^r[(1 - x)^n - (1 - x)^{n-1}];$$

whence,

$$I_n = \frac{x^{r+1}(1 - x)^n}{r + 1} + \frac{n}{r + 1} [I_{n-1} - I_n].$$

This equation may then be solved for I_n in terms of I_{n-1}:

$$I_n = \frac{x^{r+1}(1 - x)^n}{n + r + 1} + \frac{n}{n + r + 1} I_{n-1}$$

or

$$\int x^r(1 - x)^n \, dx = \frac{x^{r+1}(1 - x)^n}{n + r + 1} + \frac{n}{n + r + 1} \int x^r(1 - x)^{n-1} \, dx.$$

Now this formula may be applied recursively to express I_{n-1} in terms of I_{n-2}, I_{n-2} in terms of I_{n-3}, etc., to yield

$$I_n = \frac{x^{r+1}}{n + r + 1} \left[(1 - x)^n + \frac{n(1 - x)^{n-1}}{n + r} + \frac{n(n - 1)(1 - x)^{n-2}}{(n + r)(n + r - 1)} + \dots + \frac{n(n - 1) \dots 1}{(n + r)(n + r - 1) \dots (r + 1)} \right]$$

Sometimes it is necessary to prepare for integration by parts by some preliminary rearrangement, as we show in the following useful example.

Example A4-2g. Consider

$$I_n = \int \cos^n x \, dx$$

We write $\cos^n x = \cos^{n-1} x \cos x$, set $u = \cos^{n-1} x$, $dv = \cos x \, dx$, $v = \sin x$, to obtain

$$\begin{aligned} I_n &= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \sin^2 x \, dx \\ &= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, (1 - \cos^2 x) dx. \end{aligned}$$

Thus,

$$I_n = \cos^{n-1} x \sin x + (n - 1)[I_{n-2} - I_n].$$

Solving for I_n, we have

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n - 1}{n} I_{n-2}.$$

Since the subscript is lowered by 2 at each step we observe for n even that the recursive reduction of the integral terminates at $n = 0$ with $I_0 = \int dx = x$, and for u odd, at $n = 1$ with $I_1 = \int \cos x \, dx = \sin x$.

Often the principle use of a recurrence relation is not to obtain the formal integral in terms of elementary functions (which may not be possible) but to obtain the original integral in terms of a simpler integral.

Example A4-2h. Consider

$$I_n = \int x^n e^{-x^2} dx.$$

From $u = x^{n-1}$, $dv = x e^{-x^2} dx$, $v = -\frac{1}{2} e^{-x^2}$, we obtain

$$I_n = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{(n - 1)}{2} \int x^{n-2} e^{-x^2} dx$$

or

$$I_n = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} I_{n-2}.$$

If n is odd, the recurrence relation gives I_n in terms of elementary functions and I_1, but $I_1 = -\frac{1}{2} e^{-x^2}$ is elementary and I_n is formally integrable in terms of elementary functions. If n is even, then the integration of I_n is reduced to the integration of

$$I_0 = \int e^{-x^2}\,dx.$$

This integral is not elementary. However, it is well known and much used. In terms of the error function erf (the area under the normal probability curve) given by

$$\text{erf } x = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\,dt$$

we have

$$I_0 = \sqrt{\pi}\,\text{erf}\left(\frac{x}{\sqrt{2}}\right).$$

The common tables of the error function enable us to work with it numerically just as conveniently as the circular functions.

Exercises A4-2

1. Integrate the following.

(a) $x \sin 3x$

(b) $x \cdot 5x$

(c) $x^3 e^{-2x}$

(d) $\sqrt{x} \log ax$

(e) $\log^2 bx$

(f) $\log^3 x$

(g) $\arccos 7x$

(h) $\arctan \sqrt[3]{x}$

(i) $x \arctan x$

(j) $\dfrac{\arccos x/m}{\sqrt{x + m}}$

(k) $x \sin^2 x$

(ℓ) $x^2 \sin x$

(m) $x^2 \arcsin ax$

(n) $\cos^3 2x$

(o) $\sin^5 x$

(p) $\sin (\log ax)$

(q) $x \tan^2 x$

(r) $(\arcsin x)^2$

(s) $\sin ax \cos bx$

2. Support the geometrical interpretation of integration by parts by showing for $u = f(x)$ and $v = g(x)$ where f and g have inverses, that $u = \phi(v)$ and $v = \psi(u)$ where ϕ and ψ are inverse functions.

3. Verify as alleged after Example A4-2b that the method of the example does demonstrate the reducibility of $\int x^n f(x)dx$ to the integral of a rational function if f is any inverse circular function, or if f is the logarithmic function.

4. Establish recurrence relations for each of the following (in each case m and n are positive integers).

(a) $\int \sin^n x \, dx$

(b) $\int x^m \log^n x \, dx$

(c) $\int \sin^m x \cos^n x \, dx$

(d) $\int x^n \text{arc tan } x \, dx$

(e) $\int x^n e^{ax} \, dx$

(f) $\int x^n \text{arc sin } x \, dx$

(g) $\int \frac{1}{\sin^n x} \, dx$

(h) $\int \frac{e^x}{x^n} \, dx$

(i) $\int x^n \cos x \, dx$

(Note the difference between n odd and n even).

A4-3. Integration of Rational Functions

The problems of formal integration in the preceding sections of this appendix were often recast in the form of the problem of integrating a rational function. For a rational function there always exists a formal integral in terms of elementary functions. The formal integral is obtained by reducing the rational function to a sum of a polynomial function and functions defined by the elementary forms

$$\frac{r}{(x - c)^n} \tag{1}$$

$$\frac{px + q}{[(x - a)^2 + b^2]^n}, \qquad (b > 0). \tag{2}$$

It can be proved that such a reduction is possible, either from the Fundamental Theorem of Algebra which requires the theory of functions of a complex variable, or directly by new algebraic techniques. In either case a complete proof would take us outside the frame of this text.

The reduction of a rational function into the sum of a polynomial and terms of the form (1) and (2) is called a decomposition into partial fractions. We give one simple example.

Example A4-3a. A common case is given by the rational expression

$$\frac{1}{(x - a)(x - b)} = \frac{1}{b - a}\left(\frac{1}{x - b} - \frac{1}{x - a}\right), \qquad a \neq b. \tag{3}$$

From the decomposition (3) we immediately obtain the integral

$$\begin{aligned}\int \frac{1}{(x - a)(x - b)} &= \frac{1}{b - a}(\log(x - b) - \log(x - a)) \\ &= \frac{1}{b - a} \log \left(\frac{x - b}{x - a}\right).\end{aligned}$$

Let R be any rational function. By long division it is always possible to put $R(x)$ in the form

$$R(x) = S(x) + \frac{P(x)}{Q(x)}$$

where S, P, Q are polynomials and the degree of P is less than that of Q. Since the polynomial S is immediately integrable, we may omit it from consideration. It follows from the Fundamental Theorem of Algebra (Appendix 2) that every polynomial $Q(x)$ with real coefficients has a unique factorization of the form

$$(4)\quad Q(x) = A(x-c_1)^{n_1}(x-c_2)^{n_2}\ldots[(x-a_1)^2+b_1{}^2]^{m_1}[(x-a_2)^2+b_2{}^2]^{m_2}\ldots$$

where the c_k are the distinct real roots of Q, and $a_k \pm ib_k$, the distinct imaginary roots $(b_k > 0)$.

Now suppose that $R(x) = \frac{P(x)}{Q(x)}$ where the degree of P is less than that of Q, and that P and Q have no common factors. Then we assert that $R(x)$ is the sum of expressions of two standard forms: for each real root C, an expression of the form

$$(5)\qquad \frac{r_1}{x-c} + \frac{r_2}{(x-c)^2} + \ldots + \frac{r_n}{(x-c)^n} \qquad (r_n \neq 0)$$

where n is the multiplicity of c: for each pair of conjugate imaginary roots $a \pm ib$ an expression of the form

$$(6)\qquad \frac{p_1x+q_1}{(x-a)^2+b^2} + \frac{p_2x+q_2}{[(x-a)^2+b^2]^2} + \ldots + \frac{p_mx+q_m}{[(x-a)^2+b^2]^m}, \qquad (p_m{}^2 + q_m{}^2 \neq 0)$$

where m is their common multiplicity. We merely use this format as a guide without proof. In each particular case it can be verified directly that the decomposition obtained is correct. Once we have obtained and verified the correctness of the partial fraction decomposition we have reduced the integration problem to that of integrating the simple form (1) and (2).

Before we embark on the problem of integration let us see what is involved in the algebraic problem of obtaining the partial fraction decomposition. The first problem is to obtain the roots of the polynomial $Q(x)$. In general the roots of a polynomial cannot be obtained from the coefficients by a formula involving only rational operations and rational powers. There are such formulas for the roots of polynomials of third and fourth degree, but these formulas are generally useless. For example, the formula for the roots of a polynomial of third degree may involve complex quantities even when all three roots are real. For computational purposes it would be sufficient to estimate the roots numerically, but it is usually easier to estimate the integral directly (see Chapter 9). Nonetheless, the method of decomposition is valuable because often the factorization of $Q(x)$ is given by the conditions of the problem and often the factorization is easily obtained.

Next, we turn our attention to the problem of obtaining the partial fraction decomposition once the denominator is given in factored form.

First we consider the problem of obtaining the partial fraction decomposition of

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x - c_1)(x - c_2) \dots (x - c_n)}$$

where the roots of Q are all real and simple (of multiplicity 1) and the degree of P is less than that of Q. From the foregoing, there exist constants A_k, $(k = 1, 2, \dots, n)$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - c_1} + \frac{A_2}{x - c_2} + \dots + \frac{A_n}{x - c_n} . \tag{7}$$

For $x \neq c_1$ we obtain on multiplication by $(x - c_1)$

$$A_1 = \frac{P(x)(x - c_1)}{Q(x)} - S(x)(x - c_1) = T(x)$$

where $S(x)$ is the sum of all the partial fractions but the first. In a neighborhood of $x = c_1$ this equation states that the expression $T(x)$ defines the constant function $T : x \to A_1$. Therefore

$$\begin{aligned} A_1 &= \lim_{x \to c_1} \frac{P(x)(x - c_1)}{Q(x)} \\ &= \lim_{x \to c_1} \frac{P(x)}{(x - c_2)(x - c_3) \dots (c_1 - c_n)} \end{aligned}$$

whence,

$$A_1 = \frac{P(c_1)}{(c_1 - c_2)(c_1 - c_3) \dots (c_1 - c_n)} . \tag{8}$$

This last expression can be written tidily if we observe that since $Q(c_1) = 0$

$$\lim_{x \to c_1} \frac{Q(x)}{(x - c_1)} = \lim_{x \to c_1} \frac{Q(x) - Q(c_1)}{x - c_1} = Q'(c_1).$$

Thus $A_1 = \frac{P(c_1)}{Q'(c_1)}$. Since c_1 is simply a symbol for any one of the roots, it does not matter which for the purpose of this discussion, we have in general,

$$A_k = \frac{P(c_k)}{Q'(c_k)} . \tag{9}$$

Example A4-3b. We obtain the partial fraction decomposition of

$$\frac{x^2 + x - 1}{(x + 1)x(x - 1)} .$$

Here $P(x) = x^2 + x - 1$, $Q(x) = x^3 - x$, $Q'(x) = 3x^2 - 1$. The denominator has simple zeros at -1, 0, and 1. From

$$\frac{P(-1)}{Q'(-1)} = \frac{-1}{2} , \quad \frac{P(0)}{Q'(0)} = \frac{-1}{-1} , \quad \frac{P(1)}{Q'(1)} = \frac{1}{2} ,$$

we have

$$\frac{P(x)}{Q(x)} = - \frac{1}{2(x + 1)} + \frac{1}{x} + \frac{1}{2(x - 1)} ,$$

which is easily verified to be correct.

There are general techniques for the case of multiple real roots or imaginary roots, but in such cases it is often easier to determine the decomposition by the method of equated coefficients.*

Example A4-3c. From

$$\frac{x^3 - 1}{x(x^2 + 1)^2} = \frac{r}{x} + \frac{p_1 x + q_1}{x^2 + 1} + \frac{p_2 x + q_2}{(x^2 + 1)^2}$$

we obtain on multiplying both sides by $x(x^2 + 1)^2$

$$\begin{aligned} x^3 - 1 &= r(x^4 + 2x^2 + 1) + p_1(x^4 + x^2) + q_1(x^3 + x) + p_2 x^2 + q_2 x \\ &= (r + p_1)x^4 + q_1 x^3 + (2r + p_1 + p_2)x^2 + (q_1 + q_2)x + r, \end{aligned}$$

provided $x \neq 0$. Now the coefficients of like powers on the right and left must be equal (Exercises A4-3, No.3). Thus we obtain the equations

$$\begin{gathered} r + p_1 = 0 \\ q_1 = 1 \\ 2r + p_1 + p_2 = 0 \\ q_1 + q_2 = 0 \\ r = -1, \end{gathered}$$

from which $r = -1$, $p_1 = 1$, $q_1 = 1$, $q_2 = -1$, $p_2 = 1$. This yields

---

*Also called the method of undetermined coefficients.

$$\frac{x^3 - 1}{x(x^2 + 1)^2} = - \frac{1}{x} + \frac{x + 1}{x^2 + 1} + \frac{x - 1}{(x^2 + 1)^2}$$

which is easily verified to be correct.

Given the partial fraction decomposition of a rational function we complete the work of formal integration by showing how to integrate the standard forms (1) and (2). For (1) the integrals are already found. If $n > 1$, we have

$$\int \frac{r}{(x - c)^n}\, dx = - \frac{r}{(n - 1)(x - c)^{n-1}} + C \tag{10a}$$

and if $n = 1$, then

$$\int \frac{r}{x - 1}\, dx = r \log |x - 1| + C. \tag{10b}$$

For (2) we introduce the substitution

$$(x - a) = b \tan u \qquad \left(-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}\right),$$

where we assume $b > 0$ (compare Example A4-1b). Using $dx = \frac{b}{\cos^2 u}\, du$ we obtain

$$\int \frac{px + q}{[(x - a)^2 + b^2]^n}\, dx = \int \frac{p \tan u + pa + q}{b^{2n}[1 + \tan^2 u]^n} \frac{b}{\cos^2 u}\, du$$

$$= \frac{p}{b^{2n-1}} \int \cos^{2n-3} u \sin u \, du + \frac{pa + q}{b^{2n-1}} \int \cos^{2n-2} u \, du.$$

Of the last two integrals, the first is immediately formally integrable and the second is given by the recurrence relation of Example A4-2g. We leave as an exercise the problem of completing the integration and representing the formal integral in terms of x. The resulting integral is a sum of terms of the following types,

$$\frac{Ax + B}{[(x - a)^2 + b^2]^k} \tag{11a}$$

where k is a positive integer, $k < n$,

$$A \log [(x - a)^2 + b^2], \tag{11b}$$

$$A \arctan \frac{x - a}{b}. \tag{11c}$$

Finally, we observe that if we know the factorization of $Q(x)$ we know the form of the integral of $\frac{P(x)}{Q(x)}$ from (10) and (11). Therefore it is sufficient to differentiate this form and determine the constants by the method of equated coefficients.

Example A4-3d. Consider

$$\int \frac{x + 1}{x^2(x^2 + 4)}\, dx,$$

The integral must be of the form

$$a \log x + \frac{b}{x} + \alpha \log (x^2 + 4) + \beta \arctan \frac{x}{2} + C.$$

The derivative of this expression is

$$\frac{a}{x} - \frac{b}{x^2} + \frac{2\alpha x}{x^2 + 4} + \frac{2\beta}{x^2 + 4} = \frac{(a + 2\alpha)x^3 + (2\beta - b)x^2 + 4ax - 4b}{x^2(x^2 + 4)}$$

Since the numerator of this expression should be $x + 1$ we have on equating coefficients

$$a + 2\alpha = 0, \quad 2\beta - b = 0, \quad 4a = 1, \quad -4b = 1,$$

whence

$$a = \frac{1}{4}\,, \; b = -\frac{1}{4}\,, \; \alpha = -\frac{1}{8}\,, \; \beta = -\frac{1}{8}\,.$$

It is easy to verify that this yields the correct integral.

Exercises A4-3

1. Integrate the following

(a) $\dfrac{x + 2}{x^2 + 3x + 1}$

(b) $\dfrac{x^3}{x^2 + 3x - 10}$

(c) $\dfrac{x^3}{x^2 + 2ax + b^2}$ $(b > |a|)$

(d) $\dfrac{x^2 + \alpha x + \beta}{(x - a)(x - b)}$
(Consider the cases
$a \neq b$ and $a = b$)

(e) $\dfrac{x^2}{(x - a)(x - b)(x - c)}$ $(a \neq b \neq c)$

(f) $\dfrac{x^3 + 1}{x^3 - 1}$

(g) $\dfrac{1}{x^3 + a^2}$

(h) $\dfrac{(x + 2)^2}{x(x - 1)^2}$

(i) $\dfrac{1}{x^4 - 1}$ (ℓ) $\dfrac{x^4}{x^4 + 1}$

(j) $\dfrac{x^2}{x^4 - 1}$ (m) $\dfrac{1}{x^6 - 1}$

(k) $\dfrac{1}{x^6 + x^4}$

2. Prove from Equation (3) that if

$$Q(x) = (x - a_1)(x - a_2) \dots (x - a_n), \qquad \text{where}$$

$a_1 < a_2 < \dots < a_n$, then $\dfrac{1}{Q(x)}$ has a decomposition into partial fractions of the form

$$\frac{1}{Q(x)} = \frac{r_1}{x - a_1} + \frac{r_2}{x - a_2} + \dots + \frac{r_n}{x - a_n} .$$

3. Prove if

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$

for all but finitely many numbers x., that the coefficients of like powers on the right and left are equal; i.e., $a_k = b_k$ for $k = 0, 1, \dots, n$.

4. Verify that $\displaystyle\int \frac{px + q}{[(x - a)^2 + b^2]}\, dx$ can be expressed as the sum of terms of the forms (11a, b, c).

A4-4. Definite Integrals

In Chapter 9 and earlier sections of this appendix we addressed ourselves primarily to the problem of finding the indefinite integral of a given function. In principle, this solves the problem of evaluating any definite integral of the function. In practice, it is often desirable or necessary to evaluate a definite integral, not by formal integration, but by some other method altogether. It may be impossible to obtain an explicit representation of the indefinite integral in terms of elementary functions, yet some special symmetry may yield the value of a given definite integral effortlessly. Even if the formal expression for the indefinite integral is obtainable, the use of a symmetry condition may be a worthwhile shortcut. Often the idea of integral remains appropriate when the Riemann integral, as strictly defined, does not exist because the range or domain of the integrand may be unbounded. In these cases, we have to extend the definition of integral in a meaningful way. All these problems are treated in this section.

(1) Symmetry. Watch for symmetries; the observation that a symmetry exists often provides a direct solution to a problem or an important simplification. We have already pointed out one useful symmetry in Section 6-4.

If f is an odd function and integrable on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0. \tag{1}$$

Example A4-4a. Consider

$$I = \int_{-\pi}^{\pi} x\, e^{x^2} \sin^4 x\, dx.$$

It is hopeless to find the indefinite integral, and it is not needed, since $I = 0$.

If f is an integrable even function on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx. \tag{2}$$

Example A4-4b. Consider

$$I = \int_{-x}^{x} (a_0 + a_1 t + a_2 t^2 + \dots + a_{2n} t^{2n}) dt.$$

The odd powers contribute zero and for the even powers we obtain

$$\begin{aligned} I &= 2\int_{0}^{x} (a_0 + a_2 t^2 + \dots + a_{2n} t^{2n}\, dx \\ &= 2\left(a_0 x + \frac{a_2 x^3}{3} + \dots + \frac{a_{2n} x^{2n+1}}{2n+1}\right). \end{aligned}$$

Often an integral which exhibits no obvious symmetry can be transformed into a symmetric integral. This is specific for each case and no general rule for discovering such symmetries can be given.

Example A4-4c. Consider

$$I = \int_{-1}^{5} \sqrt[3]{x - 2}\, dx$$

Since the graph $y = \sqrt[3]{x - 2}$ has a center of symmetry at $x = 2$, we set $u = x - 2$ and find

$$I = \int_{-3}^{3} \sqrt[3]{u}\, du = 0.$$

Another important symmetry of a function is periodicity.

If the function f is integrable and periodic with period p, then the integrals of f over intervals of length p are all the same; i.e.,

$$\int_{a}^{a+p} f(x)dx = \int_{b}^{b+p} f(x)dx \tag{3}$$

for all a and b.

The statement is geometrically obvious. The graph $y = f(x)$ over any interval of length p represents the complete graph in the sense that the picture of the function from a to p is identical to the picture from $a + kp$ to $a + (k + 1)p$ where k is an integer. The entire graph can be thought of as a sequence of identical pictures of width p, laid end-to-end (Figure A4-4a). If a frame of width p is laid over the graph (the

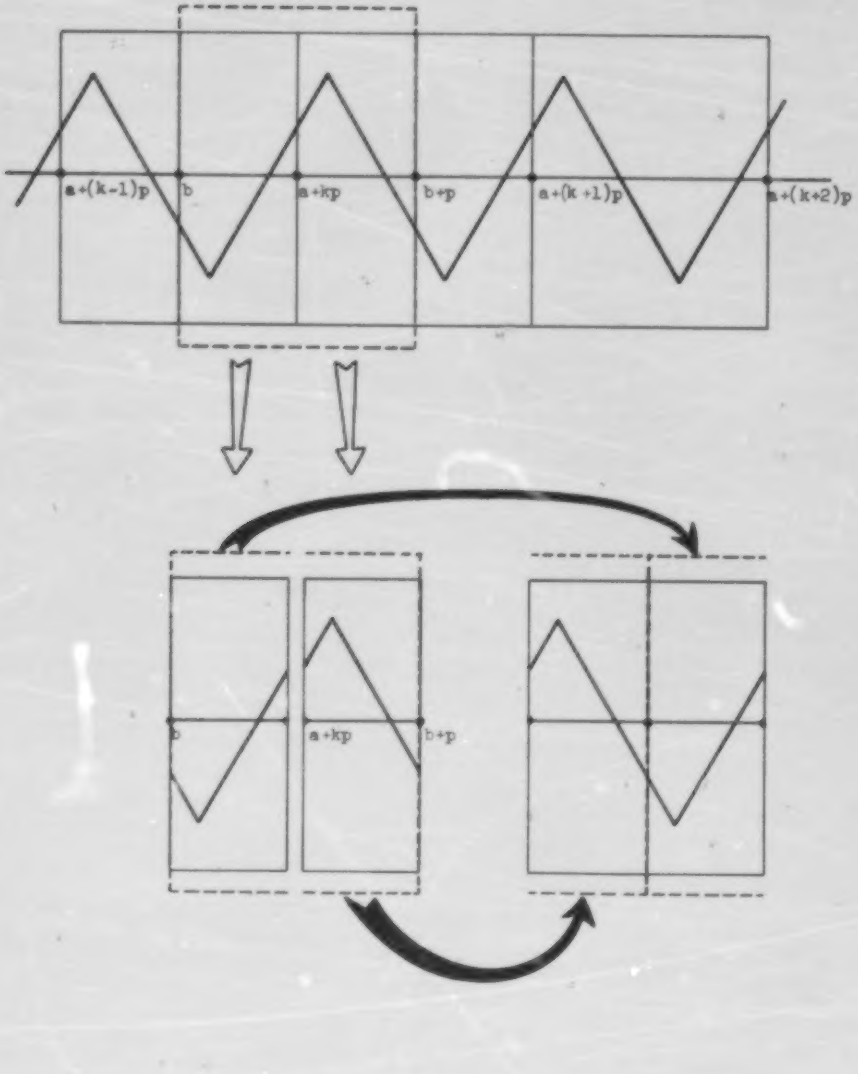

a+(k-1)p
b
a+kp
b+p
a+(k+1)p
a+(k+2)p
b
a+kp
b+p

Figure A4-4a

interval $[b, b + p]$ in the figure) then the part of the total graph within the frame may be cut along a line $a + kp$ and reassembled to form the original picture by interchanging the two piece formed by the cut. This geometrical discussion is exactly paraphrased by the analytical proof. The proof is left to Exercises A4-4, Number 12.

Example A4-4d. Consider

$$I = \int_0^{n+1/4} (a_0 + a_1 \cos 2\pi x + \dots + a_k \cos 2k\pi x)dx.$$

Since the integrand is periodic with period 1,

$$I = n\int_0^1 \sum_{\nu=0}^{k} a_\nu \cos 2\nu\pi x \, dx + \int_0^{1/4} \sum_{\nu=0}^{k} a_\nu \cos 2\nu\pi x \, dx.$$

For $\nu > 0$,

$$\int_0^1 \cos 2\nu\pi x \, dx = \frac{\sin 2\nu\pi x}{2\nu\pi}\Bigg|_0^1 = 0$$

and

$$\int_0^{1/4} \cos 2\nu\pi x \, dx = \frac{\sin \left(\frac{\nu\pi}{2}\right)}{2\nu\pi} .$$

Consequently,

$$I = \left(n + \frac{1}{4}\right) a_0 + \frac{a_1}{2\pi} - \frac{a_3}{6\pi} + \frac{a_5}{10\pi} - \dots .$$

(ii) Special reductions. The general form of a recurrence relation for a definite integral is

$$\int_a^b f_n(x)dx = g_n(x)\Bigg|_a^b + c_n\int_a^b f_{n-1}(x)dx.$$

Quite often specific problems lead to integrals for which the "boundary" term

$$g_n(x)\Bigg|_a^b = g_n(b) - g_n(a) ,$$

is zero for $n > 0$, say. If so, we immediately have

$$\int_a^b f_n(x) = c_n \, c_{n-1} \cdots c_1 \int_a^b f_0(x).$$

Thus in Example A4-2f, we could conclude at once from

$$\int x^m(1 - x)^n \, dx = \frac{x^{m+1}(1 - x)^n}{m + n + 1} + \frac{n}{n + m + 1} \int x^m(1 - x)^{n-1} \, dx$$

that

$$\int_0^1 x^m(1 - x)^n \, dx = \frac{n(n - 1) \ldots 1}{(n + m + 1)(n + m) \ldots (m + 2)} \int_0^1 x^m \, dx$$

$$= \frac{n(n - 1) \ldots 1}{(n + m + 1)(n + m) \ldots (m + 1)} .$$

Thus we obtain an important connection with the binomial coefficients:

$$\int_0^1 x^m(1 - x)^n \, dx = \left[(n + m + 1) \binom{n+m}{m}\right]^{-1}.$$

Example A4-4e. A case of special interest is

$$I_\nu = \int_0^{\pi/2} \cos^n x \, dx.$$

From the result of Example A4-2g, we have

$$I_\nu = \frac{\cos^{\nu-1} x \sin x}{\nu} \Bigg|_0^{\pi/2} + \frac{\nu - 1}{\nu} I_{\nu-2}.$$

For $\nu > 1$, this yields simply

(4) $$I_\nu = \frac{\nu - 1}{\nu} I_{\nu-2}.$$

For ν even, $\nu = 2n$, we obtain

(5a) $$I_{2n} = \frac{(2n - 1)(2n - 3) \ldots 1}{2n(2n - 2) \ldots 2} \frac{\pi}{2} .$$

For ν odd, $\nu = 2n + 1$, we obtain

(5b) $$I_{2n+1} = \frac{2n(2n - 2) \ldots 2}{(2n + 1)(2n - 1) \ldots 3} .$$

From (5a) and (5b) there can be obtained a graceful representation of $\frac{\pi}{2}$ known as Wallis's Product.* Observe that

$$\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n-1)(2n+1)} \frac{I_{2n}}{I_{2n+1}} .$$

Now, since $0 \leq \cos x \leq 1$ on $[0, \frac{\pi}{2}]$ we have $\cos^{\nu+1} x \leq \cos^{\nu} x$ for all ν so that $I_{\nu+1} \leq I_{\nu}$. It follows that $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$, and since $I_{2n-1} = \frac{2n+1}{2n} I_{2n+1}$, that

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq 1 + \frac{1}{2n} .$$

Taking limits we obtain $\lim\limits_{h \to \infty} \frac{I_{2n}}{I_{2n+1}} = 1$, whence

$$\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots$$

where by this infinite product, we mean simply

$$\lim_{n \to \infty} \left[\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n-1)(2n+1)} \right]$$

$$= \lim_{n \to \infty} \frac{1}{2n+1} \left[\frac{2^{2n} (n!)^2}{(2n)!} \right]^2$$

The verification that the two expressions in these limits are equal is left as an exercise.

---

[*] John Wallis (1616 - 1703), English.

Exercises A4-4

Evaluate the following definite integrals:

1. $\displaystyle\int_{-99}^{99} \frac{\sin^{99} \frac{x}{99}}{x^2 + (99)^2}\, dx$

2. $\displaystyle\int_0^1 x^3 e^{-3x^2}\, dx$

3. $\displaystyle\int_1^e \log^3 x\, dx$

4. $\displaystyle\int_0^{\pi/2} \sin^m x\, dx$, (m, a positive integer)

5. $\displaystyle\int_0^{\pi/2} \sin^m x \cos^m x\, dx$, (m, a positive integer)

6. $\displaystyle\int_0^{\pi/2} \frac{dx}{a + b\cos x} \qquad a > b \geq 0$

7. $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^3 x\, dx$

8. $\displaystyle\int_1^2 \frac{dx}{x + x^5}$

9. $\displaystyle\int_0^b \sqrt{b^2 - x^2}\, dx$

10. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^5 \theta + 1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, d\theta$, $a > 0$, $b > 0$

11. Compare $\displaystyle\int_0^{-a} f(x)dx$ with $\displaystyle\int_{-a}^0 f(x)dx$ when f is even or odd to derive the results (1) and (2) of the text by a method other than the one you employed for Exercises 6-4, Number 4.

12. Prove if f is integrable and periodic of period p, then for all a and b

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx.$$

13. Prove that if $n \geq 2$ then

$$.500 < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} < .524.$$

14. Prove that $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x}\, dx = \pi^2.$

15. Show $\displaystyle\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n - 1)(2n + 1)} = \frac{1}{2n + 1}\left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2.$

16. Determine the value exact to two decimal places of

$$\int_1^{e^{36.1}} \frac{\sin(\pi \log x)}{x}\,dx.$$

17. Evaluate

$$\int_{-\pi/4}^{\pi/4} \frac{t + \frac{\pi}{4}}{2 - \cos 2t}\,dt.$$

(Hint: Express the integrand as the sum of a symmetric part and an integrable part.)

Appendix 5

THE INTEGRAL FOR MONOTONE FUNCTIONS

A5-1. Introduction

Area, as we treated the idea in Chapter 7, was not defined analytically but accepted as a geometrically understood concept. We did not question the idea that a region with a curved boundary has a definite area but began with the implicit assumption that it does. Our intuition did lead us to the Fundamental Theorem of Calculus enabling us to calculate areas by finding integrals. In this appendix we shall take the concept of area arrived at intuitively and express it in precise analytical terms.

Underlying our method for determining the area of a region, there are a few elementary ideas. These ideas are commonly accepted properties of area which we postulate as the basis for the formal analytical definition of area. The area function α which associates with each region R of the plane a real number, the area of R, should satisfy the following properties.

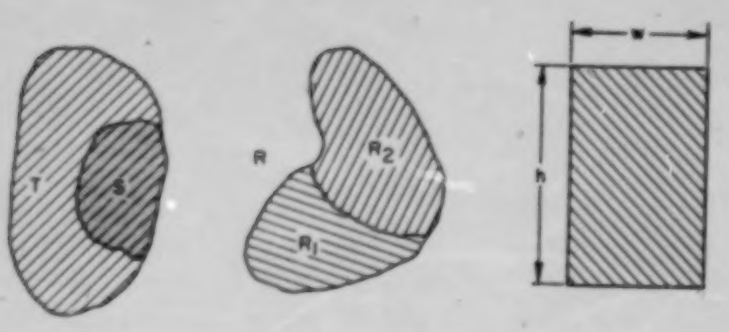
Property 1. $\alpha(R) \geq 0$

Property 2. If S and T are two regions and if S is contained in T (every point of S is also a point of T) then $\alpha(S) \leq \alpha(T)$.

Property 3. If R is the union of two nonoverlapping regions R_1 and R_2 (every point of R lies in R_1 or R_2 and only the points on their common boundary lie in both R_1 and R_2), then $\alpha(R) = \alpha(R_1) + \alpha(R_2)$.

Property 4. If R is a rectangle of height h and width w then $\alpha(R) = hw$.

Property 2 is called the order property of area and Property 3 the additive property. Properties 2-4 are illustrated in Figure A5-1a.



Property 2   Property 3   Property 4

Figure A5-1a

Exercises A5-1

1. Prove from Property 3 that if a region R is the union of n nonoverlapping regions then

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \dots + \alpha(R_n).$$

2. Show that Property 2 is actually a consequence of Property 3 given that area is nonnegative. Incorporate the notion of complementary regions.

3. (a) Using the given properties of area obtain the area of a triangle by elementary geometrical arguments

   (b) Do the same for a trapezoid.

4. If Property 4 is replaced by

   Property 4. The area of a unit square is one,
   Property 5. Congruent regions have the same area,
   show that the area of a square whose side is of length a is a^2.

5. Using the previous exercise, show that the area of a rectangle of height h and width w is hw.

A5-2. Evaluation of an Area

This section describes, in general terms, the estimation procedure of Section 7-1. Let f be a nonnegative bounded function defined on $[a,b]$. We define the standard region R under the graph of f on $[a,b]$ as the set of points bounded above by the graph of f, below by the x-axis, on the left by the vertical line $x = a$ and on the right by $x = b$; that is,

$$R = \{(x,y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$$

(Figure A5-2a). To estimate the area of R we subdivided the standard region into smaller standard regions by subdividing the base interval $[a,b]$.

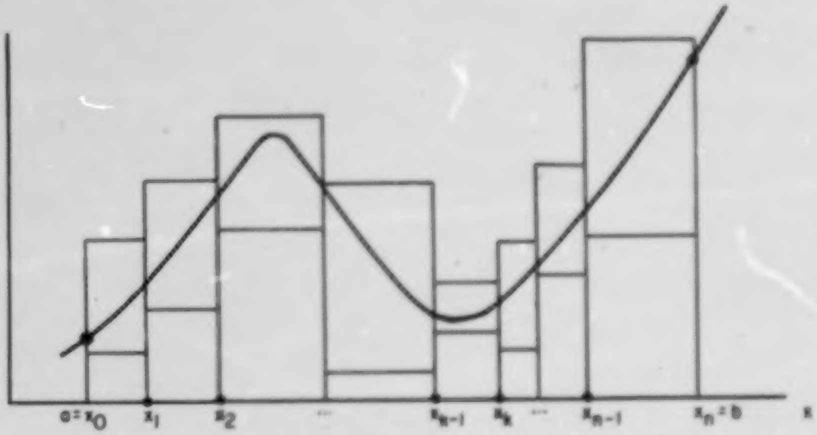


Figure A5-2a

We subdivide the interval into n parts, setting $x_0 = a$, $x_n = b$ and choosing points of subdivision $x_1, x_2, \ldots, x_{n-1}$ such that

$$x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n.^*$$

On each interval $[x_{k-1}, x_k]$, where $k = 1, 2, \ldots, n$, we have a standard region R_k where

$$R_k = \{(x,y) : x_{k-1} \leq x \leq x_k \text{ and } 0 \leq y \leq f(x)\}.$$

We then estimate the area of each subregion R_k from above and below by rectangular approximations. In each interval $[x_{k-1}, x_k]$ we obtain a lower bound m_k and an upper bound M_k for $f(x)$:

---

[*] This process is sometimes referred to as establishing the net.

$$m_k \leq f(x) \leq M_k, \qquad (x_{k-1} \leq x \leq x_k).$$

The region R_k is therefore contained in a rectangle of height M_k and, in turn contains a rectangle of height m_k on the common base $[x_{k-1}, x_k]$. We conclude from Property 2 and Property 4 (Section A5-1), that

$$m_k(x_k - x_{k-1}) \leq \alpha(R_k) \leq M_k(x_k - x_{k-1}).$$

Using the additive property, Property 3, we then have

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \dots + \alpha(R_n).$$

It follows that

$$\alpha(R) \geq m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots + m_n(x_n - x_{n-1})$$

and

$$\alpha(R) \leq M_1(x_1 - x_0) + M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1}).$$

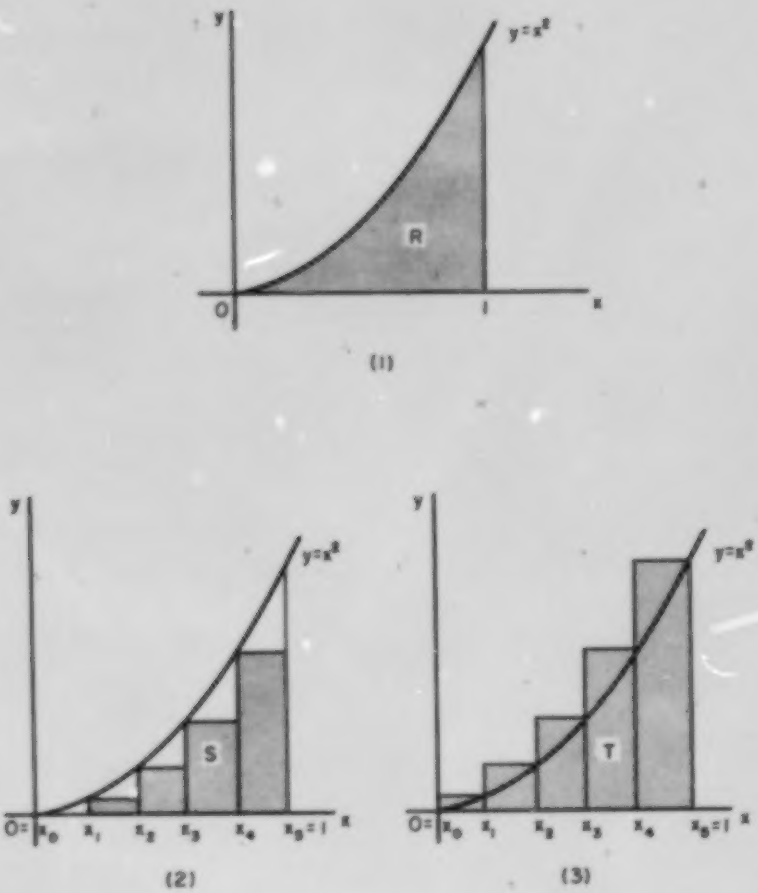
In abbreviated sum notation (Section A3-2) we have

$$\sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \alpha(R) \leq \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

Let us review this method for the function $x \to x^2$.

Consider the region R under the graph $y = x^2$ on $[0,1]$, (the shaded region in Figure A5-2b(1)). Since f is an increasing function on $[0,1]$ it will be easy to approximate $\alpha(R)$ from above and below in the manner of Section 7-1.

We use a subdivision of $[0,1]$ into n equal intervals by means of the subdivision points $x_0 = 0$, $x_1 = \frac{1}{n}$, $\dots$, $x_{n-1} = \frac{n-1}{n}$, $x_n = \frac{n}{n} = 1$. On the k-th interval of the subdivision, $x_{k-1} \leq x \leq x_k$, we have $f(x_{k-1}) \leq f(x) \leq f(x_k)$ since f is increasing.


y
y=x²
R
O
1
x
y
y=x²
S
O=x₀
x₁
x₂
x₃
x₄
x₅=1
x
y
y=x²
T
O=x₀
x₁
x₂
x₃
x₄
x₅=1
x

(1)

(2)

(3)

Figure A5-2b

We conclude that the standard region R_k based on the interval $[x_{k-1},x_k]$ contains the rectangle S_k of height $f(x_{k-1})$ and is contained in the rectangle T_k of height $f(x_k)$, both on the same base. The union of the non-overlapping rectangles S_k forms a region S which is contained within R, and the union of the rectangles T_k contains R. From the properties of area we may then obtain upper and lower estimates for the area $\alpha(R)$.

We have $\alpha(S) \leq \alpha(R) \leq \alpha(T)$, where

$$\alpha(S) = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{k=1}^{n} (k^2 - 2k + 1)$$

$$= \frac{1}{n^3} \left\{ \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} (2k - 1) \right\}$$

and

$$\alpha(T) = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1})$$

$$= \frac{1}{n^3} \sum_{k=1}^{n} k^2 .$$

We recognize the second sum in the braces within the formula for $\alpha(S)$ as the sum of an arithmetic progression, the first n odd natural numbers, whose sum is n^2. The sum $\sum_{k=1}^{n} k^2$ of the first n squares appears in both the formula for $\alpha(S)$ and that for $\alpha(T)$. A general treatment of such sums is given in Section A3-2. For this particular sum we have (Example A3-1g)

$$S_n = \sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} .$$

Consequently,

$$\alpha(S) = \frac{1}{n^3}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - n^2\right] = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} ,$$

$$\alpha(T) = \frac{1}{n^3}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right] = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} .$$

Since S is contained in R, and R is contained in T, Property 2 of area states that

$$\alpha(S) \leq \alpha(R) \leq \alpha(T),$$

or

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq \alpha(R) \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} .$$

As we increase the number of subdivisions n, both $\alpha(S)$ and $\alpha(T)$ become steadily better approximations to the number $\frac{1}{3}$, and we conclude that $\alpha(R) = \frac{1}{3}$. Formally, given any tolerance $\epsilon > 0$ we choose n to satisfy the inequality

$$\frac{1}{2n} + \frac{1}{6n^2} \leq \epsilon;$$

then $\alpha(R)$ differs from $\alpha(S)$ or $\alpha(T)$ by at most ϵ, and the estimate $\alpha(S)$ from below and $\alpha(T)$ from above differ from each other by at most 2ϵ.

Special summation techniques can be used to obtain the areas of standard regions for other functions. In Section A5-3 such summation techniques are used for the power function $x \to x^n$ and the circular function $x \to \cos x$. Often it is not convenient, sometimes not possible, to represent the area as a limit of sums which may be easily evaluated. The Fundamental Theorem of Calculus offers simpler and more general techniques but these, too, may fail. The idea of approximation is the fundamental one, and if all else fails we can always resort to obtaining approximations from above and below by the Trapezoidal Rule or Simpson's Rule to find the area of a standard region.

Exercises A5-2

1. Use the summation method to find the area of the standard region defined by

   (a) $f : x \to c,\ 0 \leq x \leq b,\ c > 0.$

   (b) $f : x \to cx,\ 0 \leq x \leq b,\ c > 0.$

   (c) $f : x \to x^2 + 2x,\ 0 \leq x \leq b.$

   (d) $f : x \to \sin(ax + b);\ 0 \leq x \leq c;$ a, b, c such that $\sin(ax + b) \geq 0$ on $[0,c]$.

   (e) $f : x \to \cos^2 x,\ 0 \leq x \leq c.$

2. Determine the area of the standard region for $f : x \to \sqrt{x}$ on $[0,1]$. (The summation encountered will be similar to the one encountered in this section.)

3. Obtain the result of Exercise 2 using only the fact that the area under the graph of $f : x \to x^2$ on $[0,1]$ is $\frac{1}{3}$, together with the basic properties of area, without resort to summation techniques.

4. Show how the upper estimating sums for $\sqrt{x}$ are related term-by-term to the lower estimating sums for x^2. (Hint: Sketch a graph of $y = x^2$. Use this graph and the y-axis to represent the standard region defined by $\sqrt{x}$.)

5. If $S_n = \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}$, show that

$$\frac{2}{3}\sqrt{n^3} < S_n < \frac{2}{3}\sqrt{n^3} + \sqrt{n}.$$

A5-3. Integration by Summation Techniques

(1) Integral of a polynomial.

In Section 7-5 we noted that integration is a linear operation, that the integral of a linear combination of functions is the same linear combination of their integrals:

$$\int_a^b [c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)]dx$$

$$= c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx + \dots + c_n\int_a^b f_n(x)dx.$$

In particular for a polynomial, we have

$$\int_a^b \sum_{r=0}^{n} c_r x^r \, dx = \sum_{r=0}^{n} c_r \int_a^b x^r \, dx.$$

In order to integrate a polynomial, then, it is sufficient to be able to integrate positive integral powers.

We have

$$\int_a^b f(x)dx = \int_c^b f(x)dx - \int_c^a f(x)dx$$

provided that f is integrable over an interval containing the points a, b, c. (See the discussion preceeding Example 7-5e.) In particular, for a polynomial we have

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx.$$

We need therefore consider only integrals of the type $\int_0^a f(x)dx$.

Consider, in particular, the integral of x^r over $[0,a]$. Since $0 \le x \le a$, the function x^r is increasing on the interval. We take a partition σ which subdivides the interval into n equal parts of length $h = \nu(\sigma) = \frac{a}{n}$. We form the upper sum U over σ using the maximum of x^r in each subinterval; thus

(1)
$$
\begin{aligned}
U &= \sum_{k=1}^{n} x_k^r (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} (kh)^r h \\
&= h^{r+1} \sum_{k=1}^{n} k^r
\end{aligned}
$$

According to Equation (4) of Section A3-2 (ii) we have

$$k^r = \frac{k^{r+1} - (k - 1)^{r+1}}{r + 1} + P(k)$$

where P is a polynomial of degree $r - 1$. It follows that

(2)
$$U = \frac{h^{r+1}}{r + 1} \sum_{k=1}^{n} [k^{r+1} - (k - 1)^{r+1}] + Q(h)$$

where

(3)
$$Q(h) = h^{r+1} \sum_{k=1}^{n} P(k)$$

and P is a polynomial of degree $r - 1$.

We recognize the sum in (2) as telescoping (Section A3-2(ii)) and obtain

$$U = \frac{h^{r+1}}{r + 1}[n^{r+1} - 0] + Q(h) = \frac{(nh)^{r+1}}{r + 1} + Q(h).$$

Since $nh = a$, we have

(4)
$$U = \frac{a^{r+1}}{r + 1} + Q(h).$$

We can show that $Q(h)$ can be made closer to zero than any given error tolerance using only that the degree of $P(k)$ is at most $r - 1$. We set

$P(k) = \sum_{i=1}^{r-1} p_i k^i$. Since $k \leq n$ it follows that

$$|P(k)| \leq \sum_{i=1}^{r-1} |p_i| k^i \leq \sum_{i=1}^{r-1} |p_i| n^i \leq \sum_{i=1}^{r-1} |p_i| n^{r-1} \leq n^{r-1} \sum_{i=1}^{r-1} |p_i|,$$

In short, we have found

$$|P(k)| \le Cn^{r-1} \tag{5}$$

where the constant C is simply the sum of the absolute values of the coefficients of $P(x)$. Entering the result of (5) in (3), we have

$$\begin{aligned} |Q(h)| &\le h^{r+1} \sum_{k=1}^{n} |P(k)| \\ &\le h^{r+1} \sum_{k=1}^{n} Cn^{r-1} \\ &\le h^{r+1} \cdot n(Cn^{r-1}) \\ &\le Ca^r h, \end{aligned} \tag{6}$$

where again we use the fact that $nh = a$. It follows at once that $\lim_{h \to 0} Q(h) = 0$.

We could also form the lower sum L over σ by taking the minimum value of x^r as lower bound in each interval $[x_r, x_{r-1}]$. In this way we could obtain a result for L similar to (4) and so prove

$$\int_0^a x^r dx = \frac{a^{r+1}}{r+1}; \tag{7}$$

the details are left to the reader.

(ii) *A cosine integral.*

Let us attempt to find the integral of $\cos x$ over $[0,a]$ where we suppose $a < \pi$ so that $\cos x$ is decreasing on the interval. We take a subdivision of the interval into n equal parts of length $h = \frac{a}{n}$. Setting

$$x_k = kh, \qquad (k=1, 2, \ldots, n),$$

we obtain a lower sum L over σ

$$L = \sum_{k=1}^{n} (\cos x_k)(x_k - x_{k-1}) = h \sum_{k=1}^{n} \cos kh \tag{1}$$

and an upper sum U over σ

$$U = h \sum_{k=1}^{n} \cos(k - 1)h$$

$$= L + h[1 - \cos a].$$

From Equation (7) of Section A3-2(ii), on setting

$$\cos \frac{n(a + 1)}{2} \sin \frac{na}{2} = \frac{1}{2}[\sin(n + \frac{1}{2})a - \sin \frac{a}{2}]$$

we obtain

$$(2) \qquad \sum_{k=1}^{n} \cos kz = u(n) - u(0) = \frac{\sin(n + \frac{1}{2})z}{2 \sin \frac{1}{2} z} - \frac{1}{2} .$$

Equation (2) permits us to evaluate the limit of the lower sum given in Equation (1):

$$\lim_{h \sim 0} L = \lim_{h \sim 0} \frac{\frac{1}{2}h}{\sin \frac{1}{2} h} \sin(a + \frac{1}{2}h) - \frac{h}{2} .$$

Using the fact that $\lim\limits_{u \sim 0} \frac{\sin u}{u} = 1$ we have

$$\lim_{h \sim 0} L = \sin a.$$

Since the difference between L and U has the limit 0, we conclude that

$$\int_0^a \cos x \, dx = \sin a.$$

Exercises A5-3

1. In subsection (i) of this section we state that it follows "at once" from the inequality (6) that

$$\lim_{h \to 0} Q(h) = 0.$$

Actually, what theorems on limits are being used?

2. Show simply, without repeating the argument of the text, that the lower sum L over σ, $L = \sum_{k=1}^{n} x_{k-1}^{r} (x_{k-1} - x_k)$ also has the limit (7).

3. Employ Equation (8) of Section A3-2(ii) to obtain $\int_0^a \sin x \, dx$ for $0 < a \leq \frac{\pi}{2}$.

A5-4. The Concept of Integral. Integrals of Monotone Functions

(i) Definition of integral.

In the computation of the area of the standard region under the graph of a bounded function f on a closed interval we gave upper and lower estimates of the area in terms of upper and lower bounds for f on each interval of a subdivision. If the function f takes on maximum and minimum values on each subinterval, as it would if f were continuous or monotone, then these would give the sharpest possible bounds. When f is continuous it may be easier to use slacker bounds than to attempt to determine the extrema. For monotone functions, however, the situation is especially simple: The extreme values on an interval are taken on at the endpoints.

We may allow f to take on negative values so that the interpretation of the upper and lower sums as upper and lower estimates of an area may not be immediate. Still these upper and lower sums may serve as upper and lower estimates for some unique number which lies below all upper estimates and above all lower estimates; if such a unique number exists it is called the integral of f over the base interval. The idea of integral has far-reaching applications, and its interpretation as area, although useful for visualizing the concept of integral, is not necessarily the most important realization of the concept.

We consider a bounded function f defined on a closed interval $[a,b]$, $a \leq b$. A subdivision of $[a,b]$ into n intervals is defined by a set of points

$$\sigma = \{x_0 , x_1 , x_2 , \dots , x_{n-1} , x_n\}$$

where $x_0 = a$, $x_n = b$ and

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n.$$

We shall call a set σ of points satisfying these requirements a partition of $[a,b]$. On the k-th subinterval $[x_{k-1},x_k]$ defined by the partition σ, let m_k be a lower bound, M_k an upper bound for $f(x)$, so that

$$m_k \leq f(x) \leq M_k$$

for all x in the subinterval. We define the lower sum over σ for the lower bounds m_k as

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

and the upper sum over σ for the upper bounds M_k as

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

If f is a nonnegative function then the lower and upper sums correspond to lower and upper estimates, respectively, for the area under the graph of f on $[a,b]$. More generally, without restricting the sign of f, we use the lower and upper sums to define the integral of f, if it exists.

> DEFINITION A5-4. Let f be defined on $[a,b]$. We say that the number I is the integral of f over $[a,b]$ if there exists just one number I such that for each choice of partitions σ_1, σ_2 and all lower sums L_1 over σ_1 and upper sums U_2 over σ_2, we have
>
> $$L_1 \leq I < U_2.$$

We raise the question of existence of such a number I because it is not immediately clear. It is possible to prove that no lower sum is greater than any upper sum. Still, there may be a gap separating the values of the upper sums from those of the lower sums. If so, there is more than one number between the lower and upper sums and the integral is not defined. On the other hand, if for each $\epsilon > 0$ it is possible to find lower and upper sums which differ by less than ϵ, there is such a number I which these lower and upper sums approximate within the error tolerance ϵ; in other words, we are able to define I as the limit of upper and lower sums. We state the principle result here as a theorem which we shall use.

> THEOREM A5-4a. Let f be a bounded function on $[a,b]$. If for every positive ϵ there exists a partition σ of $[a,b]$ and lower and upper sums L and U over σ which differ by less than ϵ, then there exists a number I which is the integral of f over $[a,b]$. Conversely, if f is integrable over $[a,b]$ then there exists a partition σ with lower and upper sums L and U such that $U - L \leq \epsilon$.

If f has an integral I over $[a,b]$ we say that f is integrable over $[a,b]$.

A proof of Theorem A5-4a requires a verification of the conditions of Definition A5-4. First we must have a demonstration that no upper sum is less than any lower sum. In that event, there exists at least one number which is both a lower bound for the set of upper sums and an upper bound for the set of lower sums (Separation Axiom). It must then be shown that there is at most one number I between the upper and lower sums. This follows from the existence of an upper and a lower sum which are closer together than any prescribed tolerance ϵ. Thus the integral is determined by a squeeze between upper and lower sums. For the details see Appendix 8.

(ii) Integrability of monotone functions.

For monotone functions we may choose m_k and M_k as function values at the endpoints of $[x_{k-1},x_k]$ and it is particularly easy to obtain an estimate of the difference between the upper and lower the error of approximations to the integral. We picture the situation in terms of the area of a standard region for a nonnegative increasing function f.

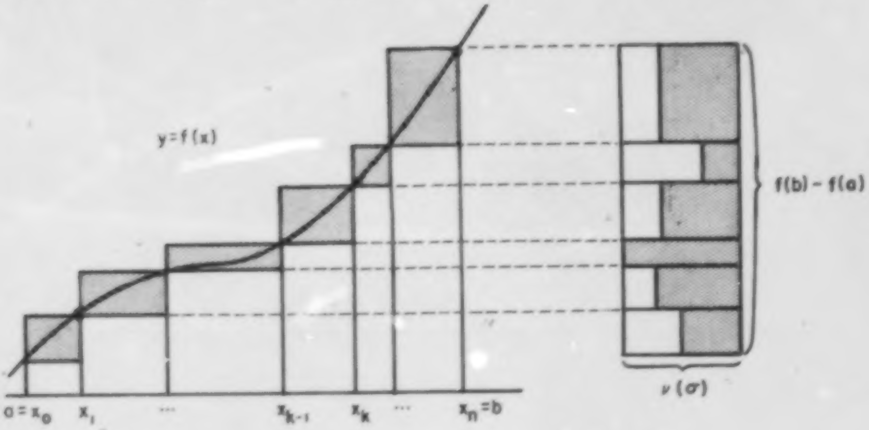


Figure A5-4a

In Figure A5-4a, the shaded rectangle over the interval $[x_{k-1},x_k]$ has height $M_k - m_k$, where $M_k = f(x_k)$ and $m_k = f(x_{k-1})$.

The total area of the shaded rectangles is the difference between the upper and lower sums for the given partition.

Since the function f is monotone we can imagine sliding these rectangles parallel to the x-axis into an arrangement with their right sides aligned. In this arrangement the rectangles are contained without overlapping in a single rectangle of height $f(b) - f(a)$ and base equal to the length of the largest interval of the subdivision. The length of the largest interval,

$$\nu(\sigma) = \max\{x_k - x_{k-1}\},$$

is a measure of the coarseness of the subdivision and is called the norm of the partition σ. We have depicted a bound on the difference between the upper and lower sums:

$$U - L \leq [f(b) - f(a)]\nu(\sigma).$$

Clearly, we can make the difference between U and L less than any error tolerance ϵ by making the subdivision fine enough, namely, by choosing σ so that

$$\nu(\sigma) \leq \frac{\epsilon}{f(b) - f(a)} .$$

Since the area I must then lie in the interval of length at most ϵ between U and L its value cannot differ from either by more than ϵ and we have satisfied the condition of Theorem A5-4a.

Although we have obtained the last result by a geometrical argument we can obtain the same result analytically with ease. We now prove: a finite monotone function on a closed interval is integrable.

THEOREM A5-4b. If f is monotone on $[a,b]$, then f is integrable over $[a,b]$.

Proof: We show that for each positive ϵ it is possible to find a partition σ of $[a,b]$ for which the difference between the upper and lower sums on the partition can be made less than ϵ:

$$U - L \leq \epsilon.$$

For this purpose we let M_k be the maximum and m_k the minimum of f on $[x_{k-1}, x_k]$. We shall prove that it is sufficient to use a subdivision σ with a norm satisfying

$$\nu(\sigma) \leq \frac{\epsilon}{|f(b) - f(a)|}$$

when $f(b) \neq f(a)$.

The case $f(b) = f(a)$ is trivial since the function f must then be a constant function. In this case, we have $M_k = m_k$ and

$$U - L = 0$$

for all subdivisions σ.

We consider the case of a weakly increasing function f (the weakly decreasing case is similar). The maximum and minimum on $[x_{k-1}, x_k]$ are given by the endpoint values

$$M_k = f(x_k) \quad \text{and} \quad m_k = f(x_{k-1}).$$

Summing over the intervals of the subdivision we have

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1})$$

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}).$$

Consequently,

$$U - L = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})](x_k - x_{k-1})$$

$$\leq \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]\nu(\sigma)$$

$$\leq \nu(\sigma) \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})].$$

We observe that

$$\sum_{k=1}^{n} f(x_k) = f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)$$

and

$$\sum_{k=1}^{n} f(x_{k-1}) = f(x_0) + f(x_1) + \ldots + f(x_{n-1})..$$

Subtracting the second of these sums from the first, we have

$$\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})] = f(x_n) - f(x_0) = f(b) - f(a);$$

consequently,

$$U - L \leq \nu(\sigma)[f(b) - f(a)].$$

To make the difference less than ϵ we need only choose $\nu(\sigma)$ as indicated above. We have satisfied the condition of Theorem A5-4a and it follows that f is integrable over $[a,b]$.

(iii) Riemann sums. Notation.

We have employed a method for defining area by approximation from above and below and extended our approach to define the more general concept of integral. This method has the great advantage of logical simplicity in the derivation of properties of the integral.

A more direct method, but one which requires somewhat more complicated argument, is to utilize values of the function in the intervals of a subdivision, instead of upper and lower bounds for approximating the area. Thus, for a function f defined on $[a,b]$ and a partition $\sigma = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ we introduce sums of the form

$$R = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{1}$$

where ξ_k is any value in the subinterval $[x_{k-1},x_k]$. These are called Riemann sums*. For a general Riemann sum the rectangle over $[x_{k-1},x_k]$ will usually not include all of the standard region under the graph and will usually include some region above the curve (Figure A5-4b) so that there will be a partial cancellation of errors. Since $m_k \leq f(\xi_k) \leq M_k$, no matter how ξ_k is chosen, we see that the Riemann sums are sandwiched between the upper and lower sums

$$L \leq R \leq U.$$

---

*After Bernhard Riemann, a German mathematician of the early 19th century, a pioneer in the careful study of the concept of integral and in other important areas.

If f has an integral I, we can therefore approximate I by Riemann sums. In fact, the approximation to I by Riemann sums can be kept within any prescribed tolerance of error for every sufficiently fine subdivision σ and corresponding choice of ξ_k. We shall then have determined the integral as a new kind of limit, a limit of Riemann sums:

$$\lim_{\nu(\sigma)\to 0} R.$$

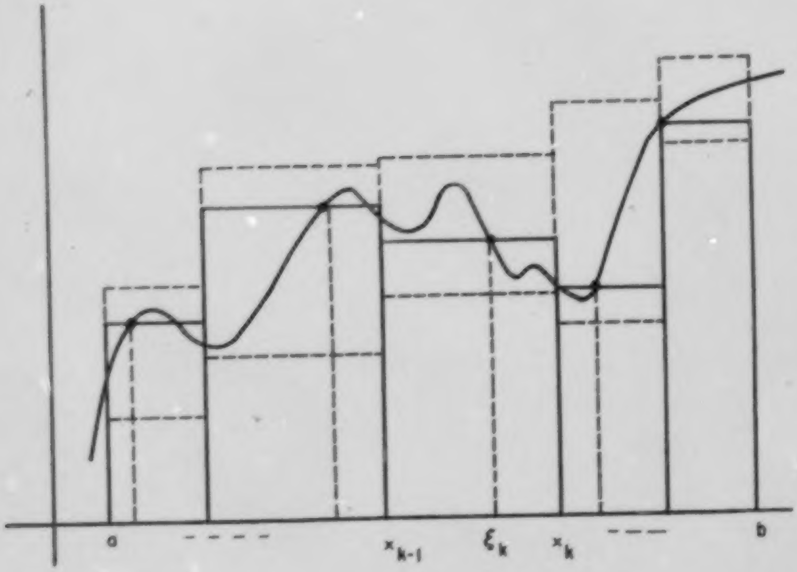


Figure A5-4b

It is natural to suppose that if this limit of Riemann sums exists, then so does the integral I of Definition A5-4, and to suppose that the two are the same. This is not an obvious proposition, but it is true. These remarks are summarized in the following theorem.

THEOREM A5-4c. The value I is the integral of f over $[a,b]$, in the sense of Definition A5-4, if and only if it is the limit of Riemann sums,

$$I = \lim_{\nu(\sigma)\to 0} R.$$

The proof will be found in Appendix 8.

The integral I of f over [a,b] is usually written in the elegant notation of Leibniz. In Leibnizian notation, the Riemann sum (1) is written

$$R = \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

where Δx_k represents the difference $x_k - x_{k-1}$. In representing the integral Leibniz used a form reminiscent of the Riemann sums,

$$I = \int_a^b f(x)\,dx.$$

Although, as we have seen, the Leibnizian notation for integral nicely complements the Leibnizian notation for derivative, it stems from conceptions which are difficult to make precise. In the thinking of Leibniz and most of the early users of the calculus, the integral sign $\int$ which is an elongated Roman "S" is a special summation symbol which replaces the corresponding Greek symbol "Σ". The integral $\int_a^b f(x)dx$ was thought of as the sum of the areas of the infinite set of "rectangles" having "infinitesimal" or "immeasurably small" base dx and height f(x) for $a \leq x \leq b$ (the Roman "d" in "dx" replaces the Greek "Δ" of the finite Riemann sum).

Exercises A5-4

1. By using upper and lower sum estimates evaluate the integral of each function f over the indicated interval.
   - (a) $f(x) = 2 - x^2$ $\quad 0 \le x \le 1$
   - (b) $f(x) = x$ $\quad 1 \le x \le 2.5$
   - (c) $f(x) = \frac{5}{2}$ $\quad 2.5 \le x \le 3$
   - (d) $f(x) = 5 - x$ $\quad 3 \le x \le 5$

2. (a) Find the minimum and the maximum values of $f(x) = 2 + 2x - x^2$ on the interval $[0,1]$, and use them to find two numbers respectively below and above the value of $\int_0^1 f(x)\, dx$.

   (b) Check your result by evaluating the integral.

3. Find upper and lower sums differing by less than .1 for the area under the graph of $f : x \to \frac{1}{x}$ on $[1,2]$.

4. Evaluate each of the following integrals, using upper and lower sum estimates.
   - (a) $\int_{-1}^{1} x^3\, dx$
   - (b) $\int_{-2}^{2} |x|\, dx$
   - (c) $\int_{-1}^{1} x^2\, dx$

5. Approximate $\int_0^1 \frac{1}{1 + x^2}\, dx$ by Riemann sums.

6. A function f defined on the interval $[a,b]$ is said to be a step-function on $[a,b]$ if for some partition $\sigma = \{x_0, x_1, \dots, x_n\}$ of the interval, $f(x)$ is constant on each open subinterval (x_{k-1}, x_k), $k = 1, 2, \dots, n$. Thus $\operatorname{sgn} x$ is a step function on $[-1,1]$, where $\operatorname{sgn} x$ is defined by

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 . \end{cases}$$

Find $\int_a^b \operatorname{sgn} x\, dx$.

7. Evaluate each of the following integrals: The function $[x]$ is defined in Appendix 1.

(a) $\int_{-1}^{3} [3x + 4]\, dx$

(b) $\int_{0}^{10} \left[\frac{x}{4}\right] dx$

(c) $\int_{1}^{5} \sqrt{2[x]}\, dx$

(d) $\int_{1}^{5} \left[\sqrt{2x}\right] dx$

8. Show that $\int_{a}^{a} f(x)dx = 0$.

A5-5. Elementary Properties of Integrals

In Section 7-4 a number of elementary properties of area were interpreted in terms of the integral notation. These area properties are, in fact, simple consequences of the four properties stated in Section A5-1. Our purpose in this section is to show that indeed these properties hold for the integral as defined by Definition A5-4. We shall make considerable use of Theorem A5-4a in this discussion.

Let f and g be nonnegative functions with $f(x) \leq g(x)$ on $[a,b]$. Since the standard region under the graph of f is contained in the standard region under the graph of g (Figure A5-5a), from Property 2 of Section 6-1 the area of the former must be no greater than the area of the latter. A similar inequality holds for integrals in general.

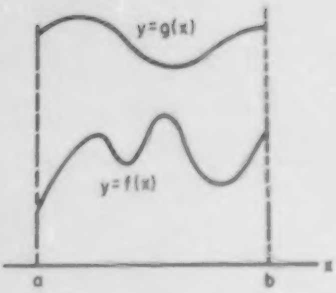


Figure 5-5a

> **THEOREM A5-5a.** If f and g are integrable and $f(x) \leq g(x)$ on $[a,b]$ then
>
> $$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. Let I denote the integral of f over $[a,b]$, and J the integral of g. We know (Theorem A5-4a) that for every positive ϵ there exist upper and lower sums U and L for g such that $U - L < \epsilon$. Since $L \leq J \leq U$ (Definition A5-4) we conclude that $U - J < \epsilon$. Thus we can find upper sums as close as desired to J. At the same time, every upper sum for J is an upper sum for I since $f(x) \leq g(x)$. We have $I \leq J$, for if we had $I > J$ we could take $\epsilon = I - J > 0$ and from $U - J < I - J$ it would follow that $U < I$, a contradiction, since U is an upper sum for I.

Consider the decomposition of the standard region over $[a,c]$ into the two standard regions over $[a,b]$ and $[b,c]$ where $a < b < c$ (see Figure A5-5b). The additive property of area (Property 3 of Section A5-1) states that the sum of the areas of the two subregions must be the area of the entire region. This corresponds to a general statement for integrals.

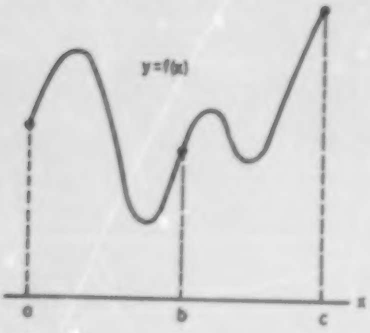


Figure A5-5b

THEOREM A5-5b. If f is integrable over $[a,c]$ then, for $a \leq b \leq c$,

(1) $$\int_a^b f(x)dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Proof. The proof of this will make use of the following result, which will be established in Appendix 8.

(2) If $a \leq c \leq d \leq b$ and f is integrable on $[a,b]$ then f is integrable on $[c,d]$.

Let us assume that f is integrable on $[a,b]$ and that $a \leq c \leq b$. Then (2) tells us that f is integrable on $[a,c]$ and on $[c,b]$, so for any $\epsilon > 0$, according to Theorem A5-4a, we can find subdivisions σ' of $[a,b]$ and σ'' of $[b,c]$ with corresponding upper and lower sums, U', L' and U'', L'' such that

$$U' - L' \leq \epsilon \text{ and } U'' - L'' \leq \epsilon.$$

Clearly, $U = U' + U''$ and $L = L' + L''$ are upper and lower sums over $[a,c]$ for the partition σ constructed by taking the two partitions σ' and σ'' together as a partition of $[a,c]$. Furthermore,

$$U - L = (U' - L') + (U'' - L'') \leq 2\epsilon.$$

For the integrals I, I', I'' over the intervals $[a,c]$, $[a,b]$, $[b,c]$, respectively, we have

$$U - I \leq 2\epsilon,\ U' - I' \leq \epsilon,\ U'' - I'' \leq \epsilon,$$

whence, for every positive ϵ,

$$\begin{aligned}|I' + I'' - I| &= |(I' - U') + (I'' - U'') - (I - U)| \\ &< \epsilon + \epsilon + 2\epsilon \\ &\leq 4\epsilon.\end{aligned}$$

It follows that $I' + I'' = I$, as we sought to prove.

In Exercise A5-4, Number 8, we noted that $\int_a^a f = 0$. By defining, for $b < a$,

$$\int_a^b f = -\int_b^a f$$

we then see that if a, b and c are any points of an interval over which f is integrable, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Linearity of integration.

For positive constants α and β integration is a linear operation:

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx,$$

for if U' and L' are upper and lower sums for f, U'' and L'' for g, it is immediate that $U = \alpha U' + \beta U''$ and $L = \alpha L' + \beta L''$ are upper and lower sums for the linear combination $\alpha f(x) + \beta g(x)$. This result does not depend on the signs of α and β as we now prove.

THEOREM A5-5c. If f and g are integrable over $[a,b]$ then any linear combination $\alpha f + \beta g$ is integrable over $[a,b]$ and

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$$

To simplify the considerations which depend on the signs of α and β we divide the proof into two parts.

Part (i). If f is integrable over $[a,b]$ then for any constant α, the function αf is integrable and

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$

Proof. Let σ be a partition of $[a,b]$ and take upper and lower sums over σ,

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1}),$$

for which $U - L < \epsilon$.

If $\alpha > 0$, then

$$\alpha U = \sum_{k=1}^{n} \alpha M_k(x_k - x_{k-1}) \quad \text{and} \quad \alpha L = \sum_{k=1}^{n} \alpha m_k(x_k - x_{k-1})$$

are upper and lower sums, respectively, for αf. It follows that

$$\alpha U - \alpha L < \alpha\epsilon$$

and hence that the difference between upper and lower sums for αf can be made less than any desired tolerance. It follows that αf is integrable. Furthermore, for the integral I of f and J of αf over $[a,b]$ we have

$$U - I < \epsilon, \quad \alpha U - J < \alpha\epsilon$$

from which it follows that

$$\begin{aligned} |J - \alpha I| &= |(J - \alpha U) + \alpha(U - I)| \\ &\leq |J - \alpha U| + \alpha|U - I| \\ &< 2\alpha\epsilon. \end{aligned}$$

Since this result holds for all positive ϵ, we conclude that $J = \alpha I$.

If $\alpha < 0$ then αU is a lower sum and αL an upper sum for αf. The proof is thus reduced to the preceding.

If $\alpha = 0$, the lemma follows trivially.

Part (ii). If f and g are integrable over $[a,b]$, then $f+g$ is integrable over $[a,b]$ and

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

We make use of an auxiliary result (from Appendix 8): Given any fixed tolerance, for any integrable function all sufficiently fine partitions have upper and lower sums closer then that tolerance. Thus for each positive ϵ, there exists some δ such that any partition σ will have an upper sum U and a lower sum L satisfying

$$|U - L| \leq \epsilon$$

whenever

$$\nu(\sigma) \leq \delta.$$

Let δ_1 and δ_2 be the controls corresponding to the given ϵ for f and g, respectively, and take $\delta = \min\{\delta_1, \delta_2\}$. Let σ be any partition with $\nu(\sigma) \leq \delta$. There then exist upper and lower sums over σ, U' and L' for f, and U'' and L'' for g such that

$$|U' - L'| \leq \epsilon \text{ and } |U'' - L''| \leq \epsilon.$$

Recall that

$$U' = \sum_{k=1}^{n} M_k'(x_k - x_{k-1}), \quad L' = \sum_{k=1}^{n} m_k'(x_k - x_{k-1})$$

and

$$U'' = \sum_{k=1}^{n} M_k''(x_k - x_{k-1}), \quad L'' = \sum_{k=1}^{n} m_k''(x_k - x_{k-1})$$

where

$$m_k' \leq f(x) \leq M_k' \text{ and } m_k'' \leq g(x) \leq M_k''.$$

Since

$$m_k' + m_k'' \leq f(x) + g(x) \leq M_k' + M_k''$$

it follows that $U = U' + U''$ is an upper sum and $L = L' + L''$ a lower sum, for $f + g$ over σ. We conclude that

$$U - L = (U' - L') + (U'' - L'') \leq 2\epsilon,$$

and it follows that $f + g$ is integrable. Furthermore, for the integrals I', I'' and I of f, g and $f + g$, respectively, we have the estimate

$$\begin{aligned} |I' + I'' - I| &= |(I' - U') + (I'' - U'') - (I - U)| \\ &\leq |I' - U'| + |I'' - U''| + |I - U| \\ &\leq \epsilon + \epsilon + 2\epsilon \\ &\leq 4\epsilon \end{aligned}$$

for each positive ϵ. It follows that $I = I' + I''$.

The derivation of Theorem A5-5c from the preceding is simple and is left as an exercise.

In one of the examples of Section 7-1 we used sums to find the area under the graph of $x \to x^2$. Employing Theorem A5-5c, we can integrate any quadratic function without resorting to estimates by upper and lower sums:

$$\int_a^b (Ax^2 + Bx + C)dx = A\int_a^b x^2\,dx + B\int_a^b x\,dx + C\int_a^b dx.$$

An immediate application of Theorem A5-5c gives the area between the graphs of two functions f and g on $[a,b]$, where $f(x) \leq g(x)$, as the integral of their difference. If $f(x) \geq 0$ as in Figure A5-5a then the area between the two graphs is simply the area of the standard region under the graph of g less the area of the standard region under the graph of f, that is,

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b [g(x) - f(x)]dx.$$

There is no reason to restrict these considerations to nonnegative functions, for if $f(x) < 0$ for some x in $[a,b]$, and m is a lower bound of $f(x)$ on $[a,b]$, we translate the x-axis vertically $|m|$ units in the negative direction so that

$$(x,y) \to (x, y + |m|).$$

In the new coordinate system the region lies between the graphs of the non-negative functions $\overline{f} : x \to f(x) + |m|$ and $\overline{g} : x \to g(x) + |m|$. (Figure A5-5c.)

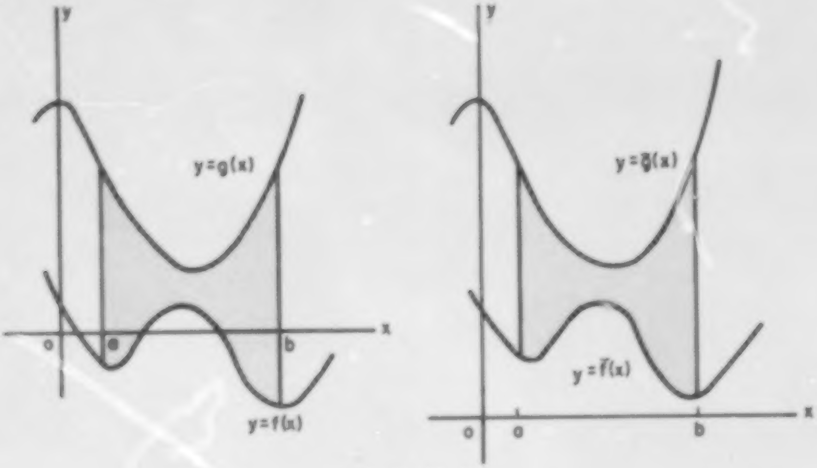

y
y=g(x)
o
a
b
x
y=f(x)
y
y=$\overline{g}$(x)
y=$\overline{f}$(x)
o
a
b
x

Figure A5-5c

Since $\overline{g}(x) - \overline{f}(x) = g(x) - f(x)$ the definition of the area of the region between the graphs of f and g as the integral of the function $g - f$ is clearly appropriate whenever $f(x) \leq g(x)$ on $[a,b]$. Thus, the area of the standard region under the graph of $F : x \to g(x) - f(x)$ on $[a,b]$ (Figure A5-5d) is equal to the area of the region between the graphs of f and g on $[a,b]$. (Figure A5-5c.)

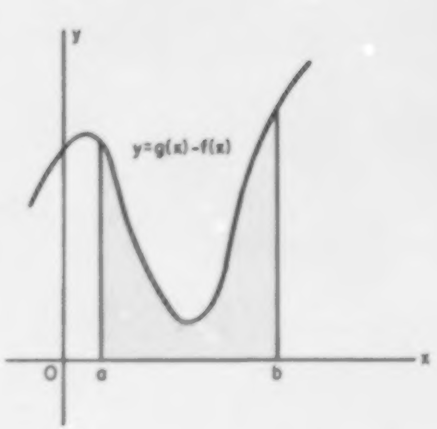

y
y=g(x)-f(x)
O
a
b
x

Figure A5-5d

Example A5-5a. Consider the area of the region between the graphs of the functions $f : x \to \cos^2 x$ and $g : x \to -\sin^2 x$ on $[0,4]$. (Figure A5-5e.)

We might attempt to represent the area of the region as the limit of sums of areas of rectangles. On the other hand, we know that the area is given by

$$\int_0^4 [f(x) - g(x)]dx,$$

since $f(x) \geq g(x)$ for all x in the interval $[0,4]$.

But
$$\int_0^4 [f(x) - g(x)]dx = \int_0^4 dx = 4,$$
since $f(x) - g(x) = \cos^2 x - (-\sin^2 x) = 1$ for all x. (The graph of $F : x \to f(x) - g(x)$ is shown in Figure A5-5f.) In conclusion we note that the area of the region shaded in Figure A5-5f is equal to the area of the region shaded in Figure A5-5e.

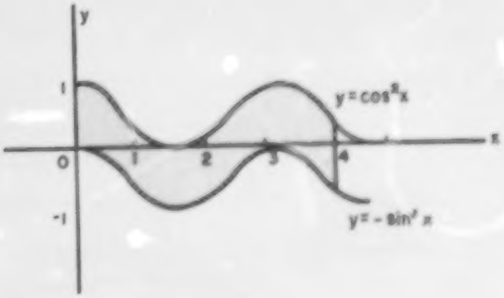


Figure A5-5e

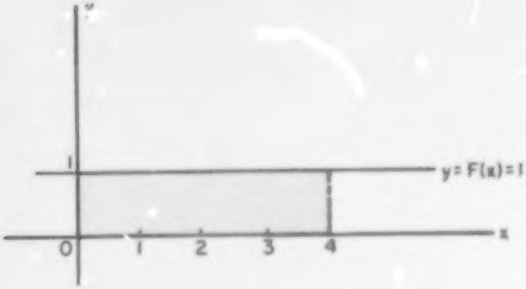


Figure A5-5f

Exercises A5-5

1. Exhibit the details of the proof of Part (i) of when $\alpha < 0$.

2. (a) If the graph of f is symmetric with respect to the origin, then f is odd. Prove that if f is odd and integrable on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0$$

   (b) If the graph of f is symmetric with respect to the y-axis, then f is even. Prove for an even function f which is integrable on $[-a,a]$ that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

   Interpret this result geometrically.

3. Prove Theorem A5-5c as a consequence of Part (i) and Part (ii). Conversely, derive these as corollaries of Theorem A5-5e.

4. Prove: If f and g are integrable where $g : x \rightarrow |f(x)|$ on $[a,b]$, then

$$\left|\int_{a}^{b} f(x)dx\right| \leq \int_{a}^{b} |f(x)|dx .$$

5. Compute the values of the given integrals using Theorem A5-5c.

   (a) $\int_{2}^{3} (3x^2 - 5x + 1)dx$

   (b) $\int_{0}^{2} (x - 1)(x + 2)dx$

   (c) $\int_{-2}^{3} (x + 2)(x - 3)dx$

6. (a) Find the area of the region below the parabola $y = x^2 - 3$ above the x-axis and between the lines $x = -3$, $x = 3$.

   (b) Find the area of the region between the graph of $f : x \rightarrow x^2 - x - 6$, the x-axis, and the lines $x = -2$, $x = 3$. First draw a rough sketch of f and indicate (by shading) the region whose area is to be computed.

7. Find all values of a for which

$$\int_{0}^{a} (x + x^2)dx = 0$$

8. Compute $\int_0^3 f(x)dx$ where

$$f(x) = \begin{cases} 2 - x^2 , & 0 \le x \le 1 \\ 5 - 4x , & 1 \le x \le 3. \end{cases}$$

9. Verify that the following property holds for $f : x \to x$

$$\int_a^b f(c - x)dx = \int_{c-b}^{c-a} f(x)dx.$$

Explain the property geometrically in terms of areas. Do you think that the property holds for other functions that are integrable? Justify your answer.

10. If a function f is periodic with period λ and integrable for all x, show that

$$\int_a^{a+n\lambda} f(x)dx = n\int_a^{a+\lambda} f(x)dx, \quad (n, \text{integer}).$$

Interpret geometrically.

11. Evaluate (without using the Fundamental Theorem of Calculus)

$$\int_0^{100\pi} (1 + \sin 2x)dx$$

12. Prove that if f is integrable on $[a,b]$ and if $f(x) \ge 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx \ge 0.$$

13. Prove that if f and g are integrable over $[a,b]$, then

$$\left|\int_a^b (g(x) - f(x))dx\right| \le \int_a^b |g(x)|dx + \int_a^b |f(x)|dx.$$

14. Let f and g be integrable and suppose that $f(x) \le g(x)$ on $[a,b]$.

(a) If the strong inequality $f(x) + \epsilon < g(x)$, for some $\epsilon > 0$, holds on any subinterval of $[a,b]$, prove the strong inequality

$$\int_a^b f(x)dx < \int_a^b g(x)dx.$$

(b) If f and g are continuous at $x = u$ in $[a,b]$ and $f(u) < g(u)$ prove that strong inequality holds as above.

15. If functions f and g are integrable, and $f(x) \leq h(x) \leq g(x)$ on $[a,b]$, does it follow that

$$\int_a^b f(x)dx \leq \int_a^b h(x)dx \leq \int_a^b g(x)dx?$$

Illustrate by an example.

16. (a) Prove the Mean Value Theorem of integral calculus: If f is continuous and integrable on $[a,b]$, then there exists a value u in the open interval (a,b) such that

$$\int_a^b f(x)dx = f(u)(b - a).$$

(b) Show that the value $f(u)$ in (a) satisfies

$$f(u) = \lim_{h \to 0} \frac{f_0 + f_1 + \dots + f_n}{n + 1}$$

where $h = \frac{(b - a)}{n}$ and $f_k = f(a + kh)$ for $k = 0, 1, 2, \dots, n$. Thus $f(u)$ can be interpreted as an extension of the idea of mean or arithmetic average to the values of a function on an interval.

17. If $\frac{a_0}{n + 1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$, show that

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1} + a_n = 0$$

has at least one root in $(0,1)$.

18. Prove that if $f(x)$ is integrable over $[a,b]$, then $|f(x)|$ is integrable over $[a,b]$. (The converse is not true. See No. 19.)

19. Suppose

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Show that if U and L are upper and lower sums for a partition of $[0,1]$ then $U \geq 1$ and $L \leq -1$. Is f integrable on $[0,1]$?

20. If f and g are integrable over $[a,b]$, then both max $\{f,g\}$ and min $\{f,g\}$ are also integrable over $[a,b]$.

21. (a) Let f and g be bounded and integrable over $[a,b]$. Prove (a) The function $f \cdot g$ is integrable over $[a,b]$;

(b) If g is bounded away from zero, then $\frac{f}{g}$ is integrable on $[a,b]$.

22. If f and g are bounded and integrable, then $\int_a^b (\alpha f(x) + \beta g(x))^2 dx$ exists and is ≥ 0 for all constant α and β.

Show from this that

$$\int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx \geq \left\{ \int_a^b f(x) \cdot g(x) dx \right\}^2 ,$$

with equality if and only if (for f and g continuous)

$$f(x) = cg(x), \quad a \leq x \leq b .$$

23. If f is integrable and its graph is convex on an interval $[0,a]$, show that *

$$\int_0^a f(x) dx \geq af(\tfrac{a}{2}).$$

Interpret geometrically.

24. Show that

$$\sqrt{(a^2 + \tfrac{1}{3})(b^2 + \tfrac{1}{3})} \geq \int_0^1 \sqrt{(x^2 + a^2)(x^2 + b^2)}\, dx .$$

25. Show that

(a) $\frac{1}{2} + \frac{3\sqrt{2}}{8} < \int_0^1 \sqrt{1 + x^3}\, dx < \frac{\sqrt{5}}{2}$.

(b) $\frac{1}{2} + \frac{\sqrt{2}}{3} > \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > \frac{2\sqrt{5}}{5}$.

---

[*] This is known as the Buniakowsky-Schwartz Inequality.

26. Find a continuously differentiable function F (i.e., F' is continuous) in $[0,1]$ which satisfies the three conditions

(a) $F(0) = 0$, $F(1) = a$,

(b) $\int_0^1 F(x)^2 dx = \frac{a^2}{3}$, and

(c) $\int_0^1 F'(x)^2 dx$ is a minimum.

Appendix 6

INEQUALITIES AND LIMITS

A6-1. Absolute Value and Inequality

The absolute value of a real number a, written $|a|$, is defined by

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0. \end{cases}$$

If we think of the real numbers in their representation on the number line, then $|a|$ is the distance between 0 and a (Figure A6-1). In general, for any real numbers a and b, the distance between a and b is $|b - a| = |a - b|$. If x lies within the span $-\epsilon \le x \le \epsilon$ where $\epsilon \ge 0$, then clearly x is no farther from the origin than ϵ and we must have $|x| \le \epsilon$. Conversely, if $|x| \le \epsilon$, then $-\epsilon \le x \le \epsilon$. It follows immediately that

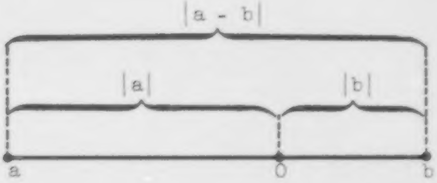


Figure A6-1

(1) $$-|x| \le x \le |x|.$$

(See Exercises A6-1, No. 13a.)

From the inequalities

$$-|a| \le a \le |a| \quad \text{and} \quad -|b| \le b \le |b|$$

we obtain

$$-(|a| + |b|) \le a + b \le |a| + |b|,$$

whence

(2) $$|a + b| \le |a| + |b|.$$

(This relation is known as the "triangle inequality.") In words, the absolute value of a sum of two terms is not greater than the sum of the absolute value of the terms. Since any sum can be built up by successive additions, the result holds in general, viz.,

$$\begin{aligned} |a + b + c| &= |(a + b) + c| \\ &\leq |a + b| + |c| \\ &\leq |a| + |b| + |c|. \end{aligned}$$

We say that y is an upper estimate for x, and that x is a lower estimate for y if $x \leq y$. In (2) we have found an upper estimate for the absolute value of the sum $a + b$. It is often useful to have a lower estimate which is better than the obvious estimate 0. Such an estimate can be obtained from (2) by the device of setting $a = x + y$ and then setting $b = -x$ and $b = -y$ in turn. We then obtain

$$|y| - |x| \leq |x + y|$$

and

$$|x| - |y| \leq |x + y|.$$

Since $||x| - |y||$ is one or the other of the values $|x| - |y|$ or $|y| - |x|$, we have

$$||x| - |y|| \leq |x + y|. \tag{3}$$

(See Exercises A6-1, No. 16.)

Special Symbols:

The symbol $\max\{r_1, r_2, \ldots, r_n\}$ denotes the largest of the numbers $r_1, r_2, \ldots, r_n$; similarly, the symbol $\min\{r_1, r_2, \ldots, r_n\}$ denotes the smallest of the numbers.

Example A6-1a

$\max\{2, 8, -3, -1\} = 8$

$\min\{2, 8, -3, -10\} = -10$

$\max\{-a, a\} = |a|$.

Exercises A6-1

1. Find the absolute value of the following numbers:

   (a) -1.75

   (b) $\frac{-\pi}{4}$

   (c) $\sin\left(\frac{-\pi}{4}\right)$

   (d) $\cos\left(\frac{-\pi}{2}\right)$

2. (a) For what real numbers x does $\sqrt{x^2} = -x$?

   (b) For what real numbers x does $|1 - x| = x - 1$?

3. Solve the equations:

   (a) $|3 - x| = 1$

   (b) $|4x + 3| = 1$

   (c) $|x + 2| = x$

   (d) $|x + 1| = |x - 3|$

   (e) $|2x + 5| + |5x + 2| = 0$

   (f) $|2x + 3| = |5 - x|$

   (g) $2|3x + 4| + |x - 2| = 1 + |3 + x|$

4. For what values of x is each of the following true? (Express your answer in terms of inequalities satisfied by x.)

   (a) $|x| \leq 0$

   (b) $|x| \neq x$

   (c) $|x| < 3$

   (d) $|x - 6| \leq 1$

   (e) $|x - 3| > 2$

   (f) $|2x - 3| < 1$

   (g) $|x - a| < a$

   (h) $|x^2 - 3| < 1$

   (i) $|(x - 2)(x - 3)| > 2$

   (j) $|x - 1| > |x - 3|$

   (k) $|x - 5| + 1 = |x + 5|$

   (ℓ) $|x - 1| + |x - 2| = 1$

   (m) $0 < |x^2 - a^2|$

   (n) $|x - a| < \delta$

   (o) $0 < |x - a| < \delta$

   (p) $|x - 1| < 2$ and $|x + 1| < \frac{3}{2}$

   (q) $|x - 1| < 2$ and $|2x - 1| < \frac{3}{2}$

   (r) $|x + y| = |x| + |y|$, for all y

   (s) $|\sin x| = 0$

   (t) $|\sin x| > \frac{\sqrt{2}}{2}$

   (u) $|1 - \frac{1}{x}| < 1$

   (v) $\sqrt{|x|} > \frac{1}{2}$

5. Sketch the graphs of the following equations:

(a) $|x - 1| + |y| = 1$

(b) $|x + y| + |x - y| = 2$

(c) $y = |x - 1| + |x - 3|$

(d) $y = |x - 1| + |x - 3| + 2|x - 4|$

(e) $y = |x - 1| + |x - 3| + 2|x - 4| + 3|x - 5|$

6. (a) Show that if $a > b > 0$, then

$$\frac{ab}{a + b} < b$$

(b) Thus, show that for positive numbers a and b, the conditon

$$\delta \le \min\{a,b\} \text{ is satisfied by } \delta = \frac{ab}{a + b}$$

7. (a) Show for positive a, b that

$$\frac{a + b}{2} < \max\{a,b\} \text{ if } a \neq b.$$

(b) Prove for all a, b that

$$\max\{a,b\} = \frac{1}{2}(a + b + |a - b|).$$

(c) Prove for all a, b that

$$\min\{a,b\} = \frac{1}{2}(a + b - |a - b|).$$

8. Show that

$$\max\{a,b\} + \max\{c,d\} \ge \max\{a + c\ ,\ b + d\}.$$

9. Show that if $ab \ge 0$, then

$$ab \ge \min\{a^2,b^2\}.$$

10. Show that if $a = \max\{a,b,c\}$, then $-a = \min\{-a,-b,-c\}$.

11. Denote $\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\right\}$ by $\min\limits_r \left(\frac{a_r}{b_r}\right)$ and similarly for max.

If $b_r > 0$, $r = 1, 2, \dots, n$, prove that

$$\min_r \left(\frac{a_r}{b_r}\right) \le \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} \le \max_r \left(\frac{a_r}{b_r}\right).$$

12. Prove that

$$\frac{1}{n} \leq \frac{1 + 2 + \ldots + n}{n^2 + (n - 1)^2 + \ldots + 2^2 + 1^2} \leq 1 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

13. (a) Prove directly from the properties of order for $\epsilon > 0$ that if $-\epsilon \leq x \leq \epsilon$ then $|x| \leq \epsilon$. Conversely, if $|x| \leq \epsilon$ then $-\epsilon \leq x \leq \epsilon$.

(b) Prove that if x is an element of an ordered field and if $|x| < \epsilon$ for all positive values ϵ, then $x = 0$.

14. (a) Prove that $|ab| = |a||b|$.

(b) Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$.

15. Prove that $|x - y| \leq |x| + |y|$.

16. Under what conditions do the equality signs hold for

$$||a| - |b|| \leq |a + b| \leq |a| + |b|?$$

17. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by x to obtain $x^2 < x$ (and, similarly, we can show that $x^3 < x^2$, $x^4 < x^3$, and so on). Use this result to show that if $0 < |x| < 1$, then $|x^2 + 2x| < 3|x|$.

18. Prove the following inequalities:

(a) $x + \frac{1}{x} \geq 2$, $x > 0$.

(b) $x + \frac{1}{x} \leq -2$, $x < 0$.

(c) $\left|x + \frac{1}{x}\right| \geq 2$, $x \neq 0$.

19. Prove: $x^2 \geq x|x|$ for all real x.

20. Show that if $|x - a| < \frac{|a|}{2}$, then

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2} \qquad \text{for all } a \neq 0.$$

21. Prove for positive a and b, where $a \neq b$, that

$$\frac{|b - a|^2}{4(a + b)} < \frac{a + b}{2} - \sqrt{ab} < \frac{|b - a|^2}{8\sqrt{ab}}.$$

A6-2. Definition of Limit of a Function

We have used the concept of limit of a function in defining derivative. At this point we present a precise formulation of the limit concept and derive the laws which govern operations with limits.

Although the concept of limit of a function is more general than the idea of derivative, our study of limits was initially motivated by the basic example of the derivative of a function ϕ as the limit of the ratio $r(x)$, which can be written

$$m = \lim_{x \to a} r(x),$$

or

$$m = \lim_{x \sim a} r(x),$$

where

$$r(x) = \frac{\phi(x) - \phi(a)}{x - a} .$$

In order to be sure that the description of the derivative as the limit of $r(x)$ makes sense we must be sure that we have an adequate set of approximations, that $r(x)$ is defined for numbers x arbitrarily close to a. Usually, the domain of r will contain an entire neighborhood of a (excluding a itself) but either for theoretical or practical reasons it is often useful to analyze the behavior of $r(x)$ on only one side of a. For example, there is a natural starting point in the motion of a rocket and it is essential to know the initial direction of the rocket in order to determine the rest of the trajectory.*

In framing the general definition of the limit of a function f at a point a we then require that we have an adequate set of approximations. Specifically, the definition may not include the value $f(a)$ among the approximations, even if it should be defined, but it must involve values $f(x)$ for x close to a. For this purpose we introduce the deleted h-neighborhood of a, that is, the set of all x for which

$$0 < |x - a| < h.$$

As the set of approximations to be used in defining the limit of f at a we take the set of values $f(x)$ for all x from the domain of f in some deleted neighborhood of a.

---

*In some texts this important case is taken care of by separate definitions of "right-sided" and "left-sided" limits. (See Exercises A6-4, No. 16.)

With these ideas in mind we are now able to express the idea of limit completely in analytical terms. If f has a limit L as x approaches a, then for any error tolerance ϵ we keep $f(x)$ within ϵ of L by restricting x to be any number from the domain of f in a sufficiently small neighborhood of a.

DEFINITION A6-2[1]. Let a be a point for which every deleted neighborhood contains points of the domain of f. The function f has the *limit* L at a if (and only if) for each positive number ϵ, there exists a positive number δ such that

$$|f(x) - L| < \epsilon$$

for every x in the domain of f which satisfies the inequality

$$0 < |x - a| < \delta.$$

We then write $\lim\limits_{x \to a} f(x) = L.$[2]

It follows from the definition of limit, since the value $f(a)$ itself does not lie in the class of approximations considered, that any function which takes on the same values as f in some deleted neighborhood of a would have the same limit at a. For example, the two functions f and g defined below have the same limit at every point a of the real axis.

$$f(x) = 1$$

$$g(x) = \begin{cases} 0, & \text{for any integer } x, \\ 1, & \text{for non-integral } x. \end{cases}$$

Although we do not rely upon pictures for our precise understanding of the concept of limit, it is desirable to have a geometrical interpretation of the idea.

Example A6-2a. The graph of the functi n

$$f : x \to 2x - 4$$

---

[1] The definition of limit can be recapitulated in terms of neighborhoods: the number L is said to be the limit of f at a if every deleted neighborhood of a contains points of the domain of f and if for each ϵ-neighborhood of L there is at least one deleted δ-neighborhood wherein f maps the points of its domain into the ϵ-neighborhood.

[2] We shall now make use of this notation, rather than $\lim\limits_{x \to a} f(x) = L$.

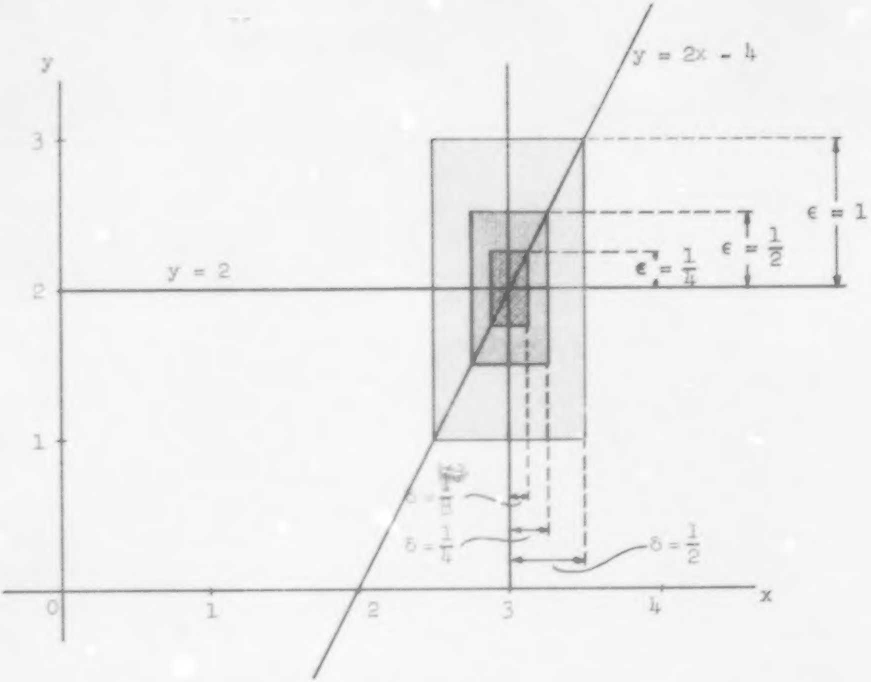
is shown in Figure A6-2a. In order to show that

$$\lim_{x \to 3} (2x - 4) = 2$$

we must show, for every $\epsilon > 0$, that there is a $\delta > 0$ so that

$$|(2x - 4) - 2| < \epsilon$$

for all x in the deleted neighborhood $0 < |x - 3| < \delta$. It is easy to see from Figure A6-2a how δ may be found.



y = 2x - 4
y
3
2
1
y = 2
$\epsilon = 1$
$\epsilon = \frac{1}{2}$
$\epsilon = \frac{1}{4}$
$\delta = \frac{1}{8}$
$\delta = \frac{1}{4}$
$\delta = \frac{1}{2}$
0
1
2
3
4
x

Figure A6-2a

Given a horizontal band of width 2ϵ centered on the line $y = 2$, we can find a vertical band of width 2δ about $x = 3$ so that the graph of f lies entirely within the rectangle where the bands overlap. From the graph we infer that for $\epsilon = 1$ we may take $\delta = \frac{1}{2}$, for $\epsilon = \frac{1}{2}$, $\delta = \frac{1}{4}$, and for $\epsilon = \frac{1}{4}$, $\delta = \frac{1}{8}$. There seems to be no obstacle to finding a δ for any ϵ, no matter how small, but we clearly cannot rely on pictures to do so. Instead, we proceed analytically. If we require $0 < |x - 3| < \delta$, then

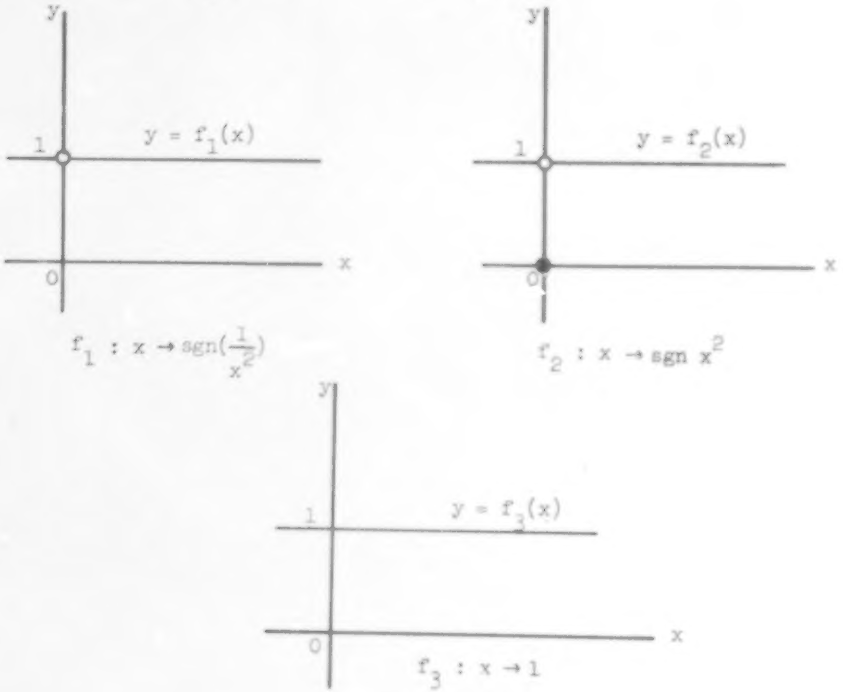
$$\begin{aligned}|f(x) - 2| &= |(2x - 4) - 2| \\ &= |2x - 6| \\ &= |2(x - 3)| \\ &= 2|x - 3| \\ &< 2\delta.\end{aligned}$$

Consequently, if we take $\delta = \frac{\epsilon}{2}$, then

$$|f(x) - 2| < \epsilon.$$

The preceding example was made especially simple to reveal the basic picture. We now explore the concept of limit in a variety of situations.

Example A6-2b. Figure A6-2b presents the graphs of the three functions given by $f_1(x) = \text{sgn}(\frac{1}{x^2})$, $f_2(x) = \text{sgn } x^2$, $f_3(x) = 1$.*



$f_1 : x \to \text{sgn}(\frac{1}{x^2})$

$f_2 : x \to \text{sgn } x^2$

$f_3 : x \to 1$

Figure A6-2b

$$^* \quad \text{sgn } \alpha = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \\ -1 & \text{if } \alpha < 0 \end{cases}$$

Observe that $x = 0$ is a point of the domains of f_2 and f_3 but not of f_1. For each of these functions we wish to consider the limit, if it exists as x approaches 0.

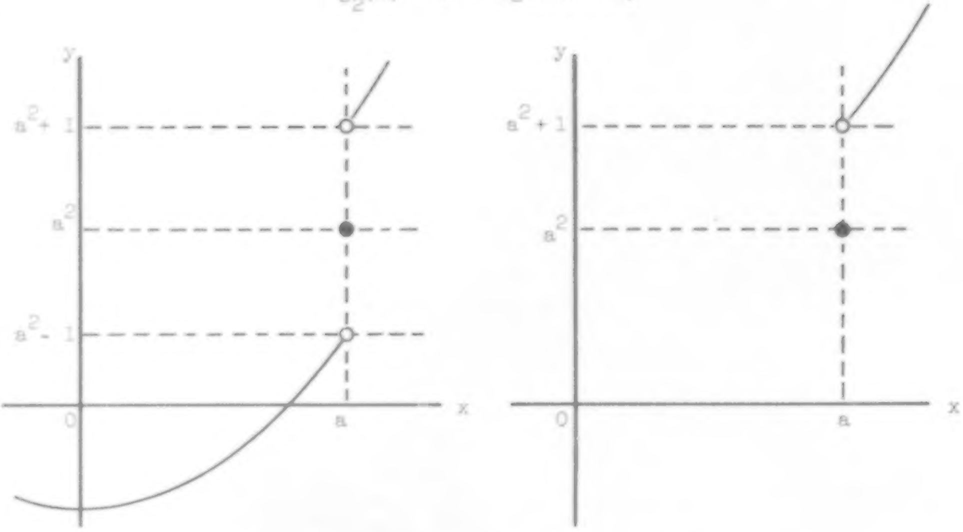
Since the three functions coincide when $x \neq 0$, and the value of the limits does not depend on how the functions are defined at $x = 0$, it is clear that all three functions have the same limit. In each case 1 is the obvious candidate for the limit. Verify that the conditions of Definition A6-2 are satisfied by $L = 1$ at $x = 0$.

Observe that there is a gap in the graphs of f_1 and f_2 at $x = 0$, and that the graph of f_3 is continuous, it has no gap. The function f_1 has a limit at $x = 0$ but is not defined there, f_2 is defined at $x = 0$ but $f_2(0)$ is not its limit, f_3 has a limit at $x = 0$ and the limit is the function value.

Example A6-2c. Figure A6-2c presents the graphs of the two functions given by

$$g_1(x) = x^2 + \operatorname{sgn}(x - a)$$

$$g_2(x) = x^2 + \operatorname{sgn}\sqrt{x - a}.$$



$g_1 : x \to x^2 + \operatorname{sgn}(x - a)$ $g_2 : x \to x^2 + \operatorname{sgn}\sqrt{x - a}$

Figure A6-2c

The function g_1 is defined for all values of x. The domain of g_2 consists only of those values of x for which $x \geq a$ and on this domain it has the same values as g_1. It seems clear from the graph that there is no single number L which is approximated by the values $g_1(x)$ as x approaches a. On the contrary, in any neighborhood of a it is possible to find values of x for which $g_1(x)$ approximates $a^2 - 1$ within any given error tolerance and other values which approximate $a^2 + 1$. Verify, then, that the conditions of Definition A6-2 cannot be satisfied, that g_1 has no limit at $x = a$.

For the function g_2, on the other hand, it appears that no matter what the error tolerance, there is a deleted neighborhood of a wherein $g_2(x)$ approximates $a^2 + 1$ within the tolerance for all x in the domain of the function. This is easily verified. In a deleted δ-neighborhood of a we have

$$g_2(x) = x^2 + 1, \quad \text{for} \quad a < x \leq a + \delta.$$

We have for the absolute error of approximation

$$\begin{aligned} |g_2(x) - (a^2 + 1)| &= |x^2 - a^2| \\ &= |x - a| \cdot |x + a| \\ &< \delta(|x| + |a|) \\ &\leq \delta[(|a| + \delta) + |a|] \\ &< \delta(2|a| + \delta). \end{aligned}$$

This absolute error can be kept within any given error tolerance ϵ by restricting x to a small enough δ-neighborhood of a. For simplicity, we first restrict ourselves to neighborhoods of radius no larger than 1. Taking $\delta \leq 1$ in the inequality above, we obtain a simpler bound on the absolute error in terms of the radius δ:

$$|g_2(x) - (a^2 + 1)| < \delta(2|a| + 1).$$

Now if we choose δ so that

$$\delta \leq \frac{\epsilon}{2|a| + 1},$$

then we have ensured that

$$|g_2(x) - (a^2 + 1)| < \epsilon,$$

namely, that the error has been kept within the tolerance ϵ. Since this is a prescription for controlling the error within any tolerance ϵ, we have accomplished our purpose and proved

$$\lim_{x \to a} g_2(x) = a^2 + 1$$

completely in the analytic terms of Definition A6-2.

Exercises A6-2

1. Show that if $0 < |x - a| < 1$, then $|x + 2a| < 1 + 3|a|$.

2. Show that if $0 < |x - a| < 1$, then $|x^3 - a^3| < (3|a^2| + 3|a| + 1)|x - a|$.

3. Show that if $0 < |x - 2| < 1$, then $\frac{1}{|x - 4|} < 1$.

   Hint: If $|x - 4| > 1$, then $\frac{1}{|x - 4|} < 1$.

4. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{1}{x^2} < \frac{4}{a^2}$.

5. Show that if $0 < |x - 1| < 1$, then $|4x + 1| < 9$ and $\left|\frac{1}{x + 2}\right| < 1$.

6. Show that if $0 < |x - 2| < 1$, then $|x + 1| < 4$ and $\left|\frac{1}{x^2 + 2x + 4}\right| < 1$.

7. Estimate how large $x^2 + 1$ can become if x is restricted to the open interval $-3 < x < 1$.

8. Use inequality properties to find a positive number M such that $0 < |x - 1| < 3$ for all x and,

   (a) $|x^2 + 2x + 4| \leq M$.

   (b) $|3x^2 - 2x + 3| \leq M$.

9. (a) Show that if $0 < |x - 3| < 1$ and $0 < |x - 3| < \frac{\epsilon}{7}$, then $|x^2 - 9| < \epsilon$.

   (b) Show that the pair of inequalities $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{7}$ (or $\delta \leq \min\{1, \frac{\epsilon}{7}\}$) is satisfied by $\delta = \frac{\epsilon}{7 + \epsilon}$.

10. Find a number $M \geq 1$ such that $\left|\frac{x + 4}{x - 4}\right| \leq M$ for all x such that $0 < |x - 2| < 1$. (See No. 3.)

11. For the given value of ϵ, find a number δ such that if $0 < |x - 3| < \delta$, $|x^2 - 9| < \epsilon$.

    (a) $\epsilon = 0.1$

    (b) $\epsilon = 0.01$

    Is your choice of δ in (b) acceptable as an answer in (a)? Explain.

12. For the following functions, find the limit L as x approaches a. For each value of ϵ, exhibit a number δ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

(a) $f(x) = 3x - 2, \quad a = \frac{1}{2}$.

(b) $f(x) = mx + b, \quad (m \neq 0)$.

(c) $f(x) = 1 + x^2, \quad a = 0$.

A6-3. Epsilonic Technique

It is conventional in discussions of approximations to a limit to use the Greek letter epsilon for the error tolerance. For this reason the subject devoted to techniques for the control of error is colloquially called epsilonics. We shall make immediate use of epsilonic technique in deriving the limit theorems which follow this section. Eventually, in applications, skill in epsilonic technique will be extremely valuable for making estimates when it is difficult to work with precise values. To develop this skill it is helpful to set up a routine pattern in which to present an epsilonic argument. We shall first describe the pattern in general and then, for several examples, carry out the proofs as indicated in the pattern.

Statement of the problem.

To prove that $\lim_{x \to a} f(x) = L$:

For each tolerance $\epsilon > 0$ obtain a control δ.

Show: if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

We have stated the problem in outline. The proof is based on Definition A6-2. We must control the error $|f(x) - L|$ within the error tolerance ϵ by restricting the values of x to a sufficiently small deleted neighborhood of a. The proof is completed by verifying for a suitable radius δ that it does give the desired degree of control. The crucial open question is, how do we choose a suitable δ?

Step 1. Simplification.

Find a $g(\delta)$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < g(\delta)$.

The idea here is to obtain an upper bound $g(\delta)$ for the absolute error where $g(\delta)$ can be held within the tolerance ϵ by taking sufficiently small values of δ. If we have $g(\delta) \leq \epsilon$, then $|f(x) - L| < g(\delta) \leq \epsilon$ and our objective is achieved. In some of the following examples the work of simplification is divided into three stages: (a) $f(x)$ is expressed in terms of $x - a$; (b) from the inequality $0 < |x - a| < \delta$ there is derived an inequality of the form $|f(x) - L| < g(\delta)$; (c) a δ is chosen for each ϵ in such a way that $g(\delta) \leq \epsilon$. In general, g is to be a simple function, one for which it is easy to find a δ such that $g(\delta) \leq \epsilon$. More typically, it will even be possible to solve for δ in the equation $g(\delta) = \epsilon$. For most of the cases in this text it is possible to obtain $g(\delta) = c\delta$ with a positive constant of proportionality c. Manipulations yielding a simple expression for $g(\delta)$ are illustrated in the Examples below.

Step 2. Choice of δ.

Choose δ so that $g(\delta) \leq \epsilon$.

This is the place where the work of simplification in Step 1 pays off. In the most typical case where $g(\delta) = c\delta$ we may choose $\delta = \frac{\epsilon}{c}$.

Steps 1 and 2 show how the solution is found. The next step is the actual proof where we verify that the solution has been found.

Step 3. Verification.

Return to the statement of the problem. From the given expression for δ deduce the conclusion.

First we try out the method in a case where no complications arise, the case of the general linear function.

Example A6-3a.

Statement of the problem.

To prove that $\lim_{x \to a} (mx + b) = ma + b$, $(m \neq 0)$.

For each $\epsilon > 0$ obtain a δ.

Show: if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Step 1. Simplification.

(a)
$$\begin{aligned} f(x) - L &= (mx + b) - (ma + b) \\ &= m(x - a). \end{aligned}$$

(b) If $|x - a| < \delta$,

$$\begin{aligned} |f(x) - L| &= |m(x - a)| \\ &= |m| \cdot |x - a| \\ &< |m|\delta. \end{aligned}$$

(c) Take $g(\delta) = |m|\delta$.

Step 2. Obtain δ.

To make $g(\delta) \leq \epsilon$, set

$$\delta = \frac{\epsilon}{|m|}$$

(allowable, since $|m| \neq 0$ by assumption).

Step 3. Verification.

Enter the result $\delta = \frac{\epsilon}{|m|}$ in the statement of the problem. The verification follows the pattern of Step 1 with one additional step:

$$
\begin{aligned}
|f(x) - L| &< |m|\delta \\
&\le |m| \frac{\epsilon}{|m|} \\
&\le \epsilon .
\end{aligned}
$$

Since there is a strong inequality in this chain, we have

$$|f(x) - L| < \epsilon .$$

In the following examples we shall omit repetitious material.

Example A6-3b.

Statement of the Problem.

To prove that $\lim\limits_{x \sim 0} \frac{1}{1 + |x|} = 1.$

For each $\epsilon > 0$ obtain a δ.

Show: if $0 < |x - 0| < \delta$, then $\left|\frac{1}{1 + |x|} - 1\right| < \epsilon .$

Step 1.

(a)

$$
\begin{aligned}
\frac{1}{1 + |x|} - 1 &= \frac{1}{1 + |x|} - \frac{1 + |x|}{1 + |x|} \\
&= \frac{-|x|}{1 + |x|} .
\end{aligned}
$$

(b) If $0 < |x - 0| < \delta$,

$$
\begin{aligned}
\left|\frac{1}{1 + |x|} - 1\right| &= \left|\frac{-|x|}{1 + |x|}\right| \\
&= \frac{|x|}{1 + |x|} \\
&< |x| \qquad \text{(since } 1 + |x| > 1) \\
&< \delta .
\end{aligned}
$$

(c) Take $g(\delta) = \delta$.

Step 2. To make $g(\delta) \le \epsilon$, set $\delta = \epsilon$.

Step 3. Set $\delta = \epsilon$ in the statement of the problem. We carry out the verification following Step 1 where we set $\delta = \epsilon$ at the last line.

The next example shows that it is not always sufficient to choose δ proportional to ϵ.

Example A6-3c.

Statement of the Problem.

To prove that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$, $(a \geq 0)$.

For each $\epsilon > 0$ obtain a δ.

Show: if $0 < |x - a| < \delta$ then $|\sqrt{x} - \sqrt{a}| < \epsilon$.

The choice $\delta = c\epsilon$, where c is a positive constant, cannot work when $a = 0$. In that case we observe that if $0 < x < \delta = c\epsilon$, then $\sqrt{x} < \sqrt{c}\,\sqrt{\epsilon}$.

We must then make $\sqrt{c}\,\sqrt{\epsilon} < \epsilon$ for all ϵ, no matter how small.

It follows that we must find a positive number c satisfying $\sqrt{c} < \sqrt{\epsilon}$ or, equivalently, $c < \epsilon$ for all positive ϵ. No such number exists; hence, $\delta = c\epsilon$ cannot work.

Step 1. From

$$|\sqrt{x} - \sqrt{a}| \leq |\sqrt{x} + \sqrt{a}| \qquad \text{(Section A6-1, Formula 3)}$$

we obtain on multiplying by $|\sqrt{x} - \sqrt{a}|$,

$$|\sqrt{x} - \sqrt{a}|^2 \leq |x - a|,$$

whence

$$|\sqrt{x} - \sqrt{a}| \leq \sqrt{|x - a|}.$$

Thus, if $0 < |x - a| < \delta$, then

$$|\sqrt{x} - \sqrt{a}| < \sqrt{\delta}.$$

Step 2. Choose $\delta = \epsilon^2$.

Step 3. Take $\delta = \epsilon^2$ in the statement of the problem. The verification is a recapitulation of Step 1 for this choice of δ.

It is often expedient to restrict δ by an auxiliary condition in Step 1. The following examples are typical.

Example A6-3d.

Statement of the Problem.

To prove that $\lim_{x \to 2} (x^3 - 5x - 1) = -3$.

For each $\epsilon > 0$ **obtain a** δ.

Show: if $0 < |x - 2| < \delta$, then $|(x^3 - 5x - 1) - (-3)| < \epsilon$.

Step 1.

(a) $$\begin{aligned} x^3 - 5x - 1 - (-3) &= x^3 - 5x + 2 \\ &= [(x - 2) + 2]^3 - 5[(x - 2) + 2] + 2 \\ &= (x - 2)^3 + 6(x - 2)^2 + 7(x - 2). \end{aligned}$$

(b) $$\begin{aligned} |x^3 - 5x - 1 - (-3)| &= |(x - 2)^3 + 6(x - 2)^2 + 7(x - 2)| \\ &= |(x - 2)[(x - 2)^2 + 6(x - 2) + 7]| \\ &= |x - 2| \cdot |(x - 2)^2 + 6(x - 2) + 7| \\ &\leq |x - 2| \cdot (|x - 2|^2 + 6|x - 2| + 7) \\ &< \delta(\delta^2 + 6\delta + 7). \end{aligned}$$

(At the last line we used $|x - 2| < \delta$.)

(c) For convenience we restrict δ by requiring $\delta \leq 1$. **Under** this condition

$$\begin{aligned} |x^3 - 5x - 1 - (-3)| &< \delta(\delta^2 + 6\delta + 7) \\ &\leq \delta(1 + 6 + 7) \\ &\leq 14\delta. \end{aligned}$$

In order to get an upper bound in the simple form $c\delta$, we put a constant bound on the second factor in $\delta(\delta^2 + 6\delta + 7)$ by restricting δ. (The particular value 1 in $\delta \leq 1$ is inessential. We could have required $\delta \leq K$ where K is any positive constant.)

Step 2. We now wish to obtain a value δ satisfying two conditions simultaneously: $\delta \leq \frac{\epsilon}{14}$ and $\delta \leq 1$. One way of satisfying these conditions is to set

$$\delta = \frac{\epsilon}{14 + \epsilon}$$

where we have chosen the denominator simply as a convenient value which is greater than either 14 or ϵ. (See Exercise A6-1, No. 6a, b.)

Step 3. Set $\delta = \frac{\epsilon}{14 + \epsilon}$ in the statement of the problem. The verification follows Step 1 through (b). In (c) we use $\delta \leq 1$ and

$$\delta \leq \frac{\epsilon}{14}$$

to obtain

$$|(x^3 - 5x - 1) - (-3)| < \epsilon.$$

Alternative Step 1.

(a) $$x^3 - 5x - 1 - (-3) = (x - 2)(x^2 + 2x - 1)$$

(b) $$\begin{aligned} |x^3 - 5x + 2| &= |x - 2| \cdot |x^2 + 2x - 1| \\ &< \delta|x^2 + 2x - 1| \\ &\leq 14\delta, \end{aligned}$$

where, at the last line, imposing the condition $\delta \leq 1$ we utilize the result $1 < x < 3$ obtained from $|x - 2| < \delta \leq 1$.

Alternative Step 2. Since we do not use the formula for δ in the verification above but only the conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{14}$, it is natural (A6-1) to set

$$\delta = \min\{\frac{\epsilon}{14}, 1\}.$$

Alternative Step 3. Set $\delta = \min\{\frac{\epsilon}{14}, 1\}$ in the statement of the problem. The verification follows alternative Step 1 above.

From the preceding example we see that we have great freedom in choosing our control δ. We can always use more stringent controls than necessary: that is, given any deleted neighborhood of $|x - a| < \delta$, so chosen that $|f(x) - L| < \epsilon$ for any x in the neighborhood, then for all x in any subset of the neighborhood and, in particular, for any smaller deleted neighborhood of a, we satisfy the same inequality. In other terms, given any δ which keeps the error within the specified tolerance, any smaller value of δ will certainly have the same effect. It follows that we may impose the condition $\delta \leq K$ where K is any convenient positive constant. Similarly, having found a δ for a particular ϵ, we know that the same δ will suffice for any larger ϵ. Hence we need concern ourselves only with these ϵ satisfying $\epsilon \leq M$, where M is any convenient positive constant.

We conclude the list of examples by applying the techniques of the outline to find some derivatives. For a given f, we set

$$r(x) = \frac{f(x) - f(a)}{x - a}, \ x \neq a.$$

Example A6-3e. $f : x \to \frac{1}{x},\ x \neq 0$

Statement of the Problem.

To prove for $a \neq 0$ that

$$\lim_{x \to a} r(x) = \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = -\frac{1}{a^2} = L.$$

For each $\epsilon > 0$ obtain a δ.

Show: if $0 < |x - a| < \delta$, then $|r(x) - L| < \epsilon$.

(Observe that $r(x)$ is not defined at $x = 0$ or $x = a$.)

Step 1.

(a)
$$\begin{aligned} r(x) - L &= \frac{\frac{1}{x} - \frac{1}{a}}{x - a} + \frac{1}{a^2} \\ &= -\frac{1}{ax} \cdot \frac{x - a}{x - a} + \frac{1}{a^2} \\ &= -\frac{1}{ax} + \frac{1}{a^2}, \qquad (x \neq a) \\ &= \frac{x - a}{a^2 x}. \end{aligned}$$

(Note that we used $|x - a| > 0$ in setting $\frac{(x - a)}{(x - a)} = 1$ for $x \neq a$.)

(b)
$$|r(x) - L| = \left|\frac{x - a}{a^2 x}\right|$$

(1)
$$< \frac{\delta}{a^2|x|}.$$

Our problem now is to obtain a constant upper bound for the factor $\frac{1}{a^2|x|} = \frac{1}{a^2|(x - a) + a|}$. It is sufficient to bound the demonimator away from 0 or to guarantee

$$|x| = |(x - a) + a| > C > 0,$$

for some number C. We have (Appendix A6-1, Formula (3))

$$|x| = |(x - a) + a| \geq |a| - |x - a|.$$

Entering $|x - a| < \delta$ in this relation, we obtain

$$|x| \geq |a| - |x - a| > |a| - \delta.$$

To obtain a constant lower bound C we restrict* $\delta \leq \frac{|a|}{2}$. In that case

$$|x| > |a| - \delta > \frac{|a|}{2} > 0$$

and $C = \frac{|a|}{2}$.

It follows from $|x - a| < \delta$ that $|x| > \frac{|a|}{2}$ and $\frac{1}{|x|} < \frac{2}{|a|}$. (See Exercises A6-1, No. 20). Consequently, from (1), we have

$$\begin{aligned} |r(x) - L| &< \frac{\delta}{a^2|x|} \\ &< \delta \frac{2}{a^2|a|} \\ &\leq \delta \frac{2}{|a|^3}. \end{aligned}$$

Step 2. The value of δ is restricted by two conditions:

$$\delta \leq \frac{|a|}{2} \quad \text{and} \quad \frac{2\delta}{|a|^3} \leq \epsilon.$$

To satisfy both conditions we take

$$\delta = \min(\epsilon \frac{|a|^3}{2}, \frac{|a|}{2}).$$

---

*Of course, in general, we could have restricted δ in any convenient way so that $\delta < |a|$. For definiteness we took

$$\delta \leq \frac{|a|}{2}\, 2.$$

Step 3. Enter the above value of δ in the statement of the problem. The verification follows the pattern of Step 1. At the last line we use

$$\delta \leq \epsilon \frac{|a|^3}{2}$$

to obtain

$$|r(x) - L| < \epsilon.$$

Example A6-3f. $f : x \to \sqrt{x}, \quad x \geq 0.$

Statement of the Problem.

To prove for $a > 0$ that $\lim\limits_{x \to a} r(x) = \lim\limits_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{x - a} = \dfrac{1}{2\sqrt{a}} = L.$

For each $\epsilon > 0$ obtain a δ.

Show: if $0 < |x - a| < \delta$, then $|r(x) - L| < \epsilon$.

(Observe that $r(x)$ is defined only for $x \geq 0$.)

Step 1.

(a)

$$\begin{aligned} r(x) - L &= \frac{\sqrt{x} - \sqrt{a}}{x - a} - \frac{1}{2\sqrt{a}} \\ &= \frac{1}{\sqrt{x} + \sqrt{a}} - \frac{1}{2\sqrt{a}} \qquad (x \neq a) \\ &= \frac{\sqrt{a} - \sqrt{x}}{2\sqrt{a}(\sqrt{x} + \sqrt{a})} \\ &= \frac{a - x}{2\sqrt{a}(\sqrt{a} + \sqrt{x})^2}. \end{aligned}$$

(Note that $\sqrt{x}$ is not defined for negative values and therefore we guarantee $0 \leq x$ by imposing the restrictions $|x - a| < a$. For this purpose we require $\delta \leq a$.)

(b)
$$
\begin{aligned}
|r(x) - L| &= \left|\frac{a - x}{2\sqrt{a}(\sqrt{a} + \sqrt{x})^2}\right| \\
&= \frac{|x - a|}{2\sqrt{a}(\sqrt{a} + \sqrt{x})^2} \\
&< \frac{\delta}{2\sqrt{a}(\sqrt{a} + \sqrt{x})^2}\,, \quad \text{(from } |x - a| < \delta\text{)}, \\
&\leq \frac{\delta}{2\sqrt{a}(\sqrt{a})^2} \qquad \text{(from } \sqrt{x} \geq 0\text{)} \\
&\leq \frac{\delta}{2(\sqrt{a})^3}\,.
\end{aligned}
$$

Step 2. Take $\delta = \min\{2(\sqrt{a})^3\epsilon,\ a\}$.

Step 3. For the above value of δ every expression used in Step 1 is defined for all x in the deleted δ-neighborhood $0 < |x - a| < \delta$. (This requires $x \neq a$ and $x \geq 0$.) The verification follows Step 1. At the last line we use $\delta \leq 2(\sqrt{a})^3\epsilon$ to obtain

$$|r(x) - L| < \epsilon.$$

> In the preceding examples we have not always followed the outline to the letter but used it only as a serviceable guide. Special difficulties are likely to appear in Step 1 and we cannot anticipate all contingencies. The only absolutely general pattern is the construction of a non-decreasing chain of expressions.
>
> $$\phi_0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n$$
>
> where $\phi_0 = |r(x) - L|$, $\phi_n = g(\delta)$ and ϕ_1, ϕ_2, $\ldots$, ϕ_{n-1} may involve both x and δ. To construct such a sequence in a particular case may require the greatest ingenuity.

In these examples we have verified that a given value L is actually the limit but have not shown how the limit L was obtained. In the next section we shall develop general theorems which will enable us to discover the value of the limit and to prove that the value is correct. Epsilonics will be necessary only to prove the theorems, not to apply them.

Exercises A6-3

1. Prove $\lim\limits_{x \to 4} (\frac{1}{2} x - 3) = -1$: obtain an upper bound $g(\delta)$ for the absolute error and find δ in terms of ϵ.

2. Give arguments that prove

   (a) $\lim\limits_{x \to a} c = c$, c any constant.

   (b) $\lim\limits_{x \to a} x = a$.

   (c) $\lim\limits_{x \to a} kx = ka$, k any constant.

   (Use the results of Example A6-3a for parts b and c.)

3. Invoke the definition directly to prove the existence of the limits in Number 2.

4. In each of the following guess the limit, and then prove that your guess is correct.

   (a) $\lim\limits_{x \to 0} \frac{1}{1 + x^2}$

   (b) $\lim\limits_{x \to 3} \frac{x^2(x - 3)}{x - 3}$

   (c) $\lim\limits_{x \to a} \frac{x^3 - a^3}{x - a}$

   (d) $\lim\limits_{x \to 1} \frac{x + 1}{x^2 + 1}$

   (e) $\lim\limits_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$

   (f) $\lim\limits_{x \to 0} \frac{x^3 - 3x - 1}{x + 2}$

   (g) $\lim\limits_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2}$

A6-4. Limit Theorems

If the epsilonic definition of limit were required in every calculation with limits, the development of the calculus would be so disjointed and so overburdened with elaborate detail that it could only be mastered by a few devoted specialists. We need and we shall derive theorems that broadly cover most of the significant calculations with limits. In the end it will only be the exceptional cases for which epsilonic techniques are necessary.

The first general results apply to rational combinations of functions, that is, expressions formed from the functions of a given set by the rational operations of addition, subtraction, multiplication, and division. If each function of the given set has a limit as x approaches a, then the limit of any rational combination of these functions is the same rational combination of the corresponding limits (with divisions by zero excluded).

There are certain special rational combinations, called linear combinations, which recur often in different contexts. It is worth distinguishing them as a class because of their importance. A linear combination is built up by addition of functions and multiplication of functions by constants. Such a linear combination can be put in the form

$$\phi : x \to \phi(x) = c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)$$

where $c_1, c_2, \dots, c_n$ are constants. In particular, a polynomial of degree less than or equal to n can be written in the form

$$\phi(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n,$$

and may therefore be thought of as a linear combination of powers $1, x, x^2, \dots, x^n$.

The evaluation of the limit of a linear combination is an instructive instance of the general method of evaluating the limits of rational combinations.

Example A6-4a.

$$\begin{aligned}\lim_{x\to 4} (6\sqrt{x} + 5x + \pi) &= \lim_{x\to 4} 6\sqrt{x} + \lim_{x\to 4} 5x + \lim_{x\to 4} \pi \\ &= (\lim_{x\to 4} 6)(\lim_{x\to 4} \sqrt{x}) + (\lim_{x\to 4} 5)(\lim_{x\to 4} x) + \lim_{x\to 4} \pi \\ &= 6 \cdot \lim_{x\to 4} \sqrt{x} + 5 \cdot \lim_{x\to 4} x + \pi.\end{aligned}$$

Note that in the example we have used three limit theorems without proof; in essence these are:

(1) The limit of the sum of two functions is the sum of the limits.

(2) The limit of the product of two functions is the product of the limits.

(3) The limit of a constant is that constant.

Consider the statement

$$\lim_{x \sim a} c = c.$$

Note that the interpretations of c on the right and left of this equation are slightly different. On the left, c stands for $f(x)$, where

$$f : x \rightarrow c$$

and on the right c is the particular value assumed by the function for each value of x. With this in mind we have

THEOREM A6-4a. For a constant function $f : x \rightarrow c$,

$$\lim_{x \sim a} f(x) = c.$$

Proof. We have

$$|f(x) - c| = |c - c| = 0 < \epsilon,$$

for every positive ϵ and every choice of δ. (The constant function is a trivial case, of course, but we include it for completeness.)

THEOREM A6-4b. If $\lim_{x \sim a} f(x) = L$, then for any constant c,

$$\lim_{x \sim a} c\, f(x) = c \lim_{x \sim a} f(x) = cL.$$

Proof. We may assume $c \neq 0$, for if $c = 0$, the problem is reduced to that of Theorem A6-4a. Given any $\epsilon > 0$, we wish to make

$$|c\, f(x) - cL| < \epsilon$$

by restricting x to a deleted neighborhood

$$0 < |x - a| < \delta.$$

From the hypothesis we know that for any ϵ^* we can find a δ^* so that if

$$0 < |x - a| < \delta^*,$$

then

$$|f(x) - L| < \epsilon^*,$$

and

$$|c\, f(x) - cL| = |c| \cdot |f(x) - L| < |c|\epsilon^*.$$

Accordingly, we choose $\epsilon^* = \frac{\epsilon}{|c|}$, obtain the appropriate value δ^* for this ϵ^*, and set $\delta = \delta^*$.

> In the following theorems we require that in some deleted neighborhood of a the domains of the functions entering the combination all coincide. This requirement eliminates nonsensical combinations such as $f(x) + g(x)$ when $f(x)$ is defined only for $x \geq a$ and $g(x)$ is defined only for $x \leq a$. The likelihood of ever making such a mistake is extremely small and therefore we do not mention this restriction on the functions explicitly in the statements or proofs of the theorems.

THEOREM A6-4c. If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M.$$

Proof. We must show that for any given $\epsilon > 0$ there is some δ such that

$$|f(x) + g(x) - (L + M)| < \epsilon$$

for all x in the common domain of f and g satisfying

$$0 < |x - a| < \delta.$$

From the hypothesis we know that for any positive ϵ_1 and ϵ_2, no matter how small, we can find δ_1 and δ_2 such that

$$|f(x) - L| < \epsilon_1 \quad \text{when} \quad 0 < |x - a| < \delta_1,$$

$$|g(x) - M| < \epsilon_2 \quad \text{when} \quad 0 < |x - a| < \delta_2.$$

But

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |f(x) - L + g(x) - M| \\ &\leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

To keep within the tolerance ϵ we can choose ϵ_1 and ϵ_2 to be any positive quantities whose sum is ϵ. For convenience, we fix

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2} .$$

Taking the appropriate values δ_1, δ_2 for these values ϵ_1, ϵ_2, we set

$$\delta = \min(\delta_1, \delta_2) .$$

For this choice of δ, whenever

$$0 < |x - a| < \delta,$$

then

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon .$$

Since a linear combination can be built up by successive operations of addition of two functions and multiplication by a constant, we obtain

Corollary. The limit of a linear combination of functions is the same linear combination of the limits of the functions; i.e., if

$$\lim_{x \sim a} f_i(x) = L_i \ , \ i = 1 \ , \ 2 \ , \ \ldots \ , \ n$$

then

$$\lim_{x \sim a} [c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x)] = c_1 \lim_{x \sim a} f_1(x) + c_2 \lim_{x \sim a} f_2(x)$$

$$+ \ldots + c_n \lim_{x \sim a} f_n(x) = c_1 L_1 + c_2 L_2 + \ldots + c_n L_n .$$

The proof is left as an exercise.

For general rational combinations we have the further operations of multiplication and division.

Example A6-4b.

$$\begin{aligned} \lim_{x \sim 4} \left[\frac{1}{x} - 2x^2 \sqrt{x}\right] &= \lim_{x \sim 4} \frac{1}{x} - (\lim_{x \sim 4} 2)(\lim_{x \sim 4} x^2)(\lim_{x \sim 4} \sqrt{x}) \\ &= \frac{1}{\lim_{x \sim 4} x} - 2(\lim_{x \sim 4} x)(\lim_{x \sim 4} x)(\lim_{x \sim 4} \sqrt{x}) \\ &= \frac{1}{4} - 2 \cdot 4 \cdot 4 \cdot 2 = -63\frac{3}{4} . \end{aligned}$$

For $\phi(x) = \frac{1}{x} - 2x^2\sqrt{x}$ let us see in detail how ϕ can be built up in simple steps. We set

$$f_1(x) = \sqrt{x},$$

$$f_2(x) = xf_1(x), \qquad \text{(multiplication)}$$

$$f_3(x) = xf_2(x), \qquad \text{(multiplication)}$$

$$f_4(x) = -2f_3(x), \qquad \text{(multiplication)}$$

$$f_5(x) = \frac{g_1(x)}{g_2(x)}, \qquad \text{(division)}$$

where

$$g_1(x) = 1$$

and

$$g_2(x) = x$$

and then,

$$\phi(x) = f_4(x) + f_5(x), \qquad \text{(addition)}.$$

It is, of course, tedious and unnecessary to decompose any rational combination into its elementary building blocks, but it is important to realize that it can be done and to know how to do it. (For example, it would be necessary to do so in writing computer programs.) In the process we have seen that to prove the general theorem concerning limits of rational combinations we now need to prove only the two special theorems for the limits of the product and quotient of two functions.

THEOREM A6-4d. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

$$\lim_{x \to a} [f(x) \cdot g(x)] = LM.$$

Proof. We wish to estimate the difference $f(x)g(x) - LM$, using the knowledge of the differences $f(x) - L$ and $g(x) - M$ given in the hypothesis. Now

$$\begin{aligned} f(x)g(x) - LM &= \Big(f(x) - L\Big)g(x) + L\Big(g(x) - M\Big) \\ &= \Big(f(x) - L\Big)\Big(g(x) - M\Big) + M\Big(f(x) - L\Big) + L\Big(g(x) - M\Big); \end{aligned}$$

hence,

$$(1)\quad |f(x)g(x) - LM| \leq |f(x) - L| \cdot |g(x) - M| + |M| \cdot |f(x) - L| + |L| \cdot |g(x) - M|.$$

From the hypothesis we know that for any positive numbers ϵ_1 and ϵ_2, there are corresponding controls δ_1 and δ_2 such that

$$|f(x) - L| < \epsilon_1 \quad \text{for} \quad 0 < |x - a| < \delta_1,$$

$$|g(x) - M| < \epsilon_2 \quad \text{for} \quad 0 < |x - a| < \delta_2.$$

Thus if we choose $\delta = \min(\delta_1, \delta_2)$, it will follow from (1) that when $0 < |x - a| < \delta$ then

$$(2)\qquad |f(x)g(x) - LM| < \epsilon_1\epsilon_2 + |M|\epsilon_1 + |L|\epsilon_2.$$

In order to keep from exceeding the tolerance ϵ we shall choose ϵ_1 and ϵ_2 so that

$$\epsilon_1\epsilon_2 + |M|\epsilon_1 + |L|\epsilon_2 \leq \epsilon;$$

this will then determine our choice of δ_1 and δ_2, and in turn that of δ. For convenience, we require that $\epsilon_1 = \epsilon_2 = \nu$ and that $\nu \leq 1$. Then

$$(3)\qquad \epsilon_1\epsilon_2 + |M|\epsilon_1 + |L|\epsilon_2 \leq \nu(1 + |L| + |M|).$$

We are now ready to choose ν and verify (3). Let

$$(4)\qquad \nu = \min\left\{1, \frac{\epsilon}{1 + |L| + |M|}\right\}.$$

Choose the corresponding δ_1 and δ_2 and let $\delta = \min(\delta_1, \delta_2)$. Then it follows from (2) and (4) when $0 < |x - a| < \delta$ that

$$|f(x)g(x) - LM| < \nu(1 + |L| + |M|) \leq \epsilon$$

as desired.

Since a polynomial $p(x)$ is a linear combination of powers, and powers are themselves products,

$$x^k = x \cdot x \ldots x \quad (k \text{ factors, } k \geq 1),$$

we can establish the following corollary.

Corollary. For any polynomial function p,

$$\lim_{x \to a} p(x) = p(a).$$

The proof of this corollary is left as an exercise (Exercises A6-4, No. 2).

To prove the limit theorem for a quotient $\frac{f(x)}{g(x)}$, it is only necessary to prove the limit theorem for a reciprocal $\frac{1}{g(x)}$. The rule for general quotients then follows from

$$\frac{f(x)}{g(x)} = f(x)\left[\frac{1}{g(x)}\right].$$

First we prove a useful preliminary result.

Lemma A6-4. If $\lim_{x \to a} g(x) = M$ and $M > 0$, then there exists a neighborhood of a where $g(x) > 0$ for x in the domain of g.

Proof. Since g has the limit M at a, there is a δ-neighborhood of a wherein $g(x)$ is closer to M than to zero:

$$|g(x) - M| < \frac{M}{2}.$$

In this neighborhood,

$$\frac{3M}{2} > g(x) > \frac{M}{2} > 0.$$

If the function ϕ has a negative limit at $x = a$ then, upon applying Lemma A6-4 to the function $-\phi$, we see at once that $\phi(x)$ is negative in some deleted neighborhood of a. As further consequences of Lemma A6-4 we have the following two corollaries.

Corollary 1. If $\lim_{x \to a} g(x) = M$ and $M \neq 0$,, then there exists a neighborhood of a where $\left|\frac{3M}{2}\right| > |g(x)| > \left|\frac{M}{2}\right|$ for x in the domain of g.

Corollary 2. A limit of a function whose values are nonnegative is nonnegative.

The proofs of these corollaries are left as exercises. (Exercises A6-4, No. 3)

THEOREM A6-4e. If $\lim_{x\to a} g(x) = M$ and $M \neq 0$, then

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}.$$

Proof. We have

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{Mg(x)}\right| \tag{2}$$

$$= \frac{|g(x) - M|}{|M| \cdot |g(x)|}$$

provided $g(x) \neq 0$. However, from Corollary 1 to Lemma A6-4 there is a δ-neighborhood of a wherein $|g(x)| > \frac{M}{2}$. Furthermore, for any ϵ^* the neighborhood can be taken so small that also

$$|g(x) - M| < \epsilon^*.$$

From (2), therefore, we have

$$\begin{aligned}\left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \frac{|g(x) - M|}{|M| \cdot |g(x)|}\\ &< \frac{\epsilon^*}{|M| \cdot \frac{|M|}{2}}\\ &\leq \frac{2\epsilon^*}{M^2}\\ &< \epsilon,\end{aligned}$$

where in the last line we have taken

$$\epsilon^* = \frac{M^2\epsilon}{2}.$$

To complete the proof we choose the value of δ appropriate to this ϵ^*.

Corollary 1. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ where $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Corollary 2. If p and q are polynomials, and if $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} .$$

In connection with these corollaries, we observe that if $\lim_{x \to a} g(x) = 0$, the quotient $\frac{f(x)}{g(x)}$ may still have a limit. Under these conditions, $\lim_{x \to a} f(x) = 0$ is a necessary but not sufficient condition for existence of $\lim_{x \to a} \frac{f(x)}{g(x)}$. The primary example is the derivative of a function expressed as the limit of a ratio for which the numerator and denominator both approach zero. It is not possible to make any general statement about the existence of the limit for such cases; it is possible that $\lim_{x \to a} f(x) = 0$ and yet that the limit of the quotient does not exist (for example, $\lim_{x \to 0} \frac{x}{x^2}$). (See Exercises A6-4f, Nos. 14 and 15.)

In estimating $\lim_{x \to a} f(x)$ we can often bound f below and above by functions g and h which have limits as x approaches a. In that case we expect that the limit of f is bounded below and above by the limits of g and h. This result is a direct consequence of the following theorem.

THEOREM A6-4f. If $f(x) \leq g(x)$ in some deleted neighborhood of a, and $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $L \leq M$.

Proof. Since $g(x) - f(x)$ is nonnegative it follows that

$$\lim_{x \to a} [g(x) - f(x)] = M - L \geq 0.$$

(Theorem A6-4c and Corollary 2 to Lemma A6-4.)

Corollary 1. [Sandwich Theorem.] If

$$h(x) \leq f(x) \leq g(x)$$

in some deleted neighborhood of a, and if

$$\lim_{x \to a} h(x) = K \text{ and } \lim_{x \to a} g(x) = M,$$

then, if $\lim_{x \to a} f(x)$ exists,

$$K \leq \lim_{x \to a} f(x) \leq M.$$

Corollary 2. [Squeeze Theorem.] If $h(x) \leq f(x) \leq g(x)$ in some deleted neighborhood of a and if

$$\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = M,$$

then

$$\lim_{x \to a} f(x) = M.$$

Exercises A6-4

1. Prove the corollary to Theorem A6-4c.

2. Prove the corollary to Theorem A6-4d.

3. Prove the corollaries to Lemma A6-4.

4. Prove the corollaries to Theorem A6-4e.

5. Find the following limits, giving at each step the theorem on limits which justifies it.

   (a) $\lim_{x \to 3} (2 + x)$

   (b) $\lim_{x \to -1} (5x - 2)$

   (c) $\lim_{x \to 0} \left(\frac{a}{1 + |x|} - b\sqrt{|x|}\right)$, where a and b are constants.

   (d) $\lim_{x \to a} (x^3 + ax^2 + a^2x + a^3)$, where a is constant.

6. Find the following limits, giving at each step the theorem which justifies it.

   (a) $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$ (b) $\lim_{x \to 3} \frac{x^2 - 9}{x^3 - 27}$

7. Find $\lim_{x \to 1} \frac{x^n - 1}{x - 1}$, for n a positive integer. Verify first that

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \dots + x + 1, \qquad (x \neq 1).$$

8. Determine whether the following limits exist and, if they do exist, find their values.

   (a) $\lim_{x \to 1} \frac{1 + \sqrt{x}}{1 - x}$

   (b) $\lim_{x \to a} (x^n - a^n)$; n is a positive integer, a is constant.

   (c) $\lim_{x \to -1} \frac{\sqrt{2 + x} + 1}{x + 1}$

   (d) $\lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{x^2 + x - 2}$

   (e) $\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}$

9. Using the algebra of limits show that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ if and only if $\lim\limits_{x \to a} \dfrac{f(x) - f(a) - L(x - a)}{|x - a|} = 0$.

10. Assume $\lim\limits_{x \to 0} \sin x = 0$ and $\lim\limits_{x \to 0} \cos x = 1$. Find each of the following limits, if the limit exists, giving at each step the theorem on limits which justifies it.

    (a) $\lim\limits_{x \to 0} \sin^3 x$

    (b) $\lim\limits_{x \to 0} \tan x$

    (c) $\lim\limits_{x \to 0} \sin 2x$

    (d) $\lim\limits_{x \to 0} \dfrac{\sin x}{\tan x}$

    (e) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x}$

    (f) $\lim\limits_{x \to 0} \dfrac{\cos 2x}{\cos x + \sin x}$

11. (a) Prove Corollary 1 to Theorem A6-4f.

    (b) Prove Corollary 2 to Theorem A6-4f.
    (Hint: Prove $\lim\limits_{x \to a} f(x)$ exists.)

12. For what integral values of m and n does $\lim\limits_{x \to -a} \dfrac{x^m + a^m}{x^n + a^n}$ exist? Find the limit for these cases.

13. Prove that if $\lim\limits_{x \to a} f(x) = 0$ and $g(x)$ is bounded in a neighborhood of $x = a$, then $\lim\limits_{x \to a} f(x) \cdot g(x) = 0$.

14. (a) Verify that if $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ exists and if $\lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} f(x) = 0$.

    (b) Describe functions f and g for which $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ yet the limit of their quotient does not exist.

15. Prove that if $\lim\limits_{x \to a} g(x) = 0$ and $\lim\limits_{x \to a} f(x)$ does not exist, then the limit of the quotient $\dfrac{f(x)}{g(x)}$ does not exist.

16. The right-hand limit at a point $P(p, f(p))$ of a function is the limit of the function at the point P for a right-hand domain $(p, p + \delta)$. Similarly, for the left-hand limit, the domain is restricted to $(p - \delta, p)$. We denote them, symbolically, by $\lim\limits_{x \to p^+} f(x)$ and $\lim\limits_{x \to p^-} f(x)$ respectively. In particular, $\lim\limits_{x \to 2^+} [x] = 2$, $\lim\limits_{x \to 2^-} [x] = 1$. Determine the indicated limits, if they exist, of the following:

(a) $\lim\limits_{x \to 2^+} \dfrac{[x]^2 - 4}{x^2 - 4}$

(b) $\lim\limits_{x \to 2^-} \dfrac{[x]^2 - 4}{x^2 - 4}$

(c) $\lim\limits_{x \to 3^+} \left(x - 2 + [2 - x] - [x]\right)$

(d) $\lim\limits_{x \to 3^-} \left(x - 2 + [2 - x] - [x]\right)$

(e) $\lim\limits_{x \to 0^+} \left(\dfrac{x}{a}\left[\dfrac{b}{x}\right] - \dfrac{b}{x}\left[\dfrac{x}{a}\right]\right)$, $a > 0$, $b > 0$

(f) $\lim\limits_{x \to 0^-} \left(\dfrac{x}{a}\left[\dfrac{b}{x}\right] - \dfrac{b}{x}\left[\dfrac{x}{a}\right]\right)$, $a > 0$, $b > 0$

(g) $\lim\limits_{x \to 0^+} \dfrac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2}$

Appendix 7

CONTINUITY THEOREMS

A7-1. Completeness of the Real Number System. The Separation Axiom

Simple algebraic and order properties do not alone serve to define the real number system; the rational numbers satisfy the same properties and so do other systems. Although no physical measurement requires anything more than the rational numbers, they are not adequate for either geometry or analysis. For example, the hypotenuse of a right triangle with legs of unit length has the irrational length $\sqrt{2}$; thus the Pythagorean Theorem would not be true if lengths were measured by rational values alone. In the rational field the concept of infinite decimal would be limited to terminating and periodic decimals; an infinite decimal like $0.101100111000\ldots$ with chains of ones and zeros of increasing length is uninterpretable in the rational field. The system of rational numbers has theoretical gaps, but the real number system is complete in that real numbers are adequate to represent all the points on a line (lengths), and all infinite decimals. At the same time, it is possible to represent any real number by a point on a line or an infinite decimal; in fact, we use the concepts of point on the number line or infinite decimal as synonymous with real number.

The completeness of the real number system, its lack of theoretical gaps, is a consequence of a geometrically plausible axiom.

The Separation Axiom. If A and B are non-empty sets of real numbers for which every number in A is less than or equal to each number in B, then there is a real number s which separates A and B; that is, for each $x \in A$ and $y \in B$ we have $x \leq s \leq y$.

In geometrical terms, if no point of a set A lies to the right of any point of a set B, then there is a point s such that all points of A (but s, should it happen to be a point of A) lie to the left of s, and all points of B (but s, if $s \in B$) lie to the right of s (see Figure A7-1a).

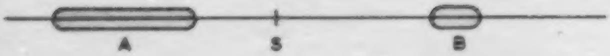


Figure A7-1a

A simple example of two sets satisfying the separation axiom is given by

$$A = \{x : x \leq -1\}, \quad B = \{y : y \geq 1\}.$$

Clearly, any number s in the interval $[-1,1]$ serves to separate these sets.

If two sets are separated by an entire interval, as in the preceding example, then it is possible to find a rational separation number s, because every interval on the number line contains rational points. The interesting cases are those for which there are elements of the two sets A and B closer together than any given positive distance. Gaps in the system of rational numbers can be exhibited as failures of the separation axiom for such sets. For example, let A be the set of positive rational numbers α satisfying $\alpha^2 < 2$, and let B be the set of positive rational numbers β satisfying $\beta^2 > 2$. It is possible to find rational values α and β closer together than any stated tolerance (see Exercises A7-1, No. 18) but a separation number s would have to satisfy $s^2 = 2$ and no rational number has that property (Exercise A7-1, No. 3c). We can define $\sqrt{2}$ as the unique real number which separates A and B. In fact, any real number can be defined as a separation number for suitable classes of rationals. More generally, it will be convenient for some purposes to determine a real number as the unique separation number for two sets by the criterion of the following lemma.

Lemma A7-1. Consider two sets of real numbers A and B such that $x \leq y$ for each $x \in A$ and each $y \in B$. If for every positive ϵ there exist $\alpha \in A$ and $\beta \in B$ such that $\beta - \alpha < \epsilon$, then the number s separating A and B is unique. Conversely, if there is just one separation number s, then for every positive ϵ there exist α and β with $\beta - \alpha < \epsilon$.

Proof. Let s and t be separation points for A and B. Given ϵ, $\alpha \in A$, and $\beta \in B$ such that $\beta - a < \epsilon$, it follows from the fact that s and t lie between α and β (Figure A7-1b) that $|s - t| < \epsilon$. Since this

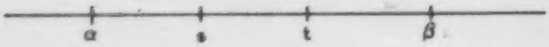


Figure A7-1b

is true for every positive ϵ it follows that $|s - t| = 0$ and hence that $s = t$ (see Exercises A7-1, No. 13b).

For the proof of the converse, let s denote the one number separating A and B. For every positive ϵ there must exist points $\alpha \in A$ and $\beta \in B$ such that

$$\alpha > s - \frac{\epsilon}{2} \text{ and } \beta < s + \frac{\epsilon}{2},$$

for should one of these inequalities fail, then we would have $s - \frac{\epsilon}{2}$ or $s + \frac{\epsilon}{2}$ as a separation number. We conclude that $\beta - \alpha < \epsilon$.

Next we derive an important consequence of the Separation Axiom.

The Least Upper Bound Principle. Let A be a set of numbers which is bounded above; i.e., there exists a value M such that $\alpha \leq M$ for all $\alpha \in A$. In the set of all upper bounds of A there is one upper bound which is smaller than any other, the least upper bound.*

Proof. Let B denote the set of upper bounds of A. The sets A and B satisfy the conditions of the Separation Axiom. It follows that there exists at least one separation number for A and B. Let s be such a separation number. Since s is a separation number it is an upper bound of A and is by definition an element of B. Since s is also a lower bound for B it is the least element of B and therefore the least upper bound of A.

The Least Upper Bound Principle is also a way of expressing the completeness of the real numbers; it is equivalent to the Separation Axiom in the sense that either may replace the axiom and that the separation property will then follow.

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*This number is also called the supremum of A and is denoted by $\sup A$. The abbreviation $\text{lub } A$ is also common.

In order to verify that the Separation Axiom and the Least Upper Bound Principle are equivalent formulations of the completeness of the real number system it is necessary to prove that in an ordered field the Least Upper Bound Principle implies the Separation Axiom. The proof is left as an exercise.

Corollary 1. If M is the least upper bound of the set A, then for each positive ϵ there exists an $\alpha \in A$ such that $\alpha > M - \epsilon$.

Corollary 2. A set of numbers which is bounded below has a greatest lower bound.

The proofs of these Corollaries are left as exercises.

There are various methods for constructing the real numbers from the rational numbers so that the usual algebraic and order properties and the Separation Axiom will hold. These will be discussed in subsequent courses.

Exercises A7-1

1. Prove Corollary 1 to the Least Upper Bound Principle.
2. Prove Corollary 2 to the Least Upper Bound Principle.
3. (a) Consider the sets A of positive rational numbers α satisfying $\alpha^2 < 2$, and B of positive rational numbers β satisfying $\beta^2 > 2$. Prove if $\alpha \in A$ and $\beta \in B$ that $\alpha < \beta$.

   (b) Show that a separation number s for the sets A and B must satisfy $s^2 = 2$; i.e., $s = \sqrt{2}$.

   (c) Prove that $\sqrt{2}$ is irrational.
4. (a) Prove for every real number a, that there is an integer n greater than a (Principle of Archimedes).

   (b) Prove that given any $\epsilon > 0$ there is an integer n such that $0 < \frac{1}{n} < \epsilon$.

5. (a) We define the infinite decimal

$$c_0.c_1c_2c_3 \ldots ,$$

where c_0 is an integer, and c_1, c_2, c_3, ..., are digits, by the number r, where

$$c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \ldots + \frac{c_n}{10^n} \leq r < c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \ldots + \frac{c_n + 1}{10^n} .$$

Show that the preceding inequality does, in fact, define a unique real number.

(b) Given a real number r we define its decimal representation recursively in terms of the integer part function $[x]$ as follows:

$$c_0 = [r]$$

$$c_n = [10^n(r - c_0 - \frac{c_1}{10} - \frac{c_2}{10^2} - \ldots - \frac{c_{n-1}}{10^{n-1}})].$$

Show that the inequality in part (a) is satisfied for this choice of c_n

Show also that decimals consisting entirely of 9's from some point on are avoided. (Thus, we obtain 2 = 2.000 ... but not 2 = 1.999 ...).

6. An infinite decimal $c_0.c_1c_2c_3 \ldots$ is said to be periodic if for some fixed value p, the period of the decimal, we have $c_{n+p} = c_n$ for all n satisfying $n \geq n_0$, where we require that p is the smallest positive integer satisfying this condition. In words, from some place on, the decimal consists of the indefinite repetition of the same p digits. Thus

$$\frac{1}{3} = 0.33333\ldots$$

$$\frac{15}{44} = 0.34090909\ldots$$

are periodic decimals. It is convenient to indicate a cycle of p digits by underlining, rather than repetition; e.g.,

$$\frac{22}{7} = 3.\underline{142857} .$$

(a) Prove that every periodic decimal represents a rational number. (Hint: Consider the decimal as a geometric progression.)

(b) Prove that every rational number has a periodic decimal representation. (A "terminating" decimal in which each place beyond a certain point is zero is considered as a special case of periodic decimals.) If $r = \frac{s}{t}$ represents a rational number given in lowest terms, find the largest possible period of the infinite decimal representation of r in terms of the denominator t.

From b we conclude that a decimal which is not periodic represents an irrational number, and conversely.

(c) Prove for every positive prime α other than 2 and 5 that there exists an integer, all of whose digits are ones, for which α is a factor; i.e., α is a factor of some number of the form

$$10^n + 10^{n-1} + 10^{n-2} + \dots + 10 + 1.$$

7. (a) Consider a polynomial with integer coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0. \qquad (a_n \neq 0)$$

Prove that if $\frac{p}{q}$ is a rational root of this polynomial given in lowest terms, then p is a factor of a_0 and q is a factor of a_n.

(b) Show that $x^3 + x + 1$ has no rational root.

(c) Prove that if $\sqrt{n}$ is rational then it is integral.

(d) Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

A7-2. The Extreme Value and Intermediate Value Theorems for Continuous Functions

In Section 8-2 we stated two theorems, which we reiterate here in more precise terms.

THEOREM 8-2a. The Intermediate Value Theorem.
Suppose f is continuous at each point of the interval $a \leq x \leq b$ and that $f(a) \neq f(b)$. If d lies between $f(a)$ and $f(b)$ then there is at least one point c between a and b such that

$$f(c) = d.$$

THEOREM 8-2b. Suppose f is continuous at each point of the interval $a \leq x \leq b$. Then there are points c and d, with $a \leq c \leq b$ and $a \leq d \leq b$ such that

$$f(d) \leq f(x) \leq f(c) \quad \text{for all } x, \quad a \leq x \leq b.$$

These two theorems will be proved in this section. Our proof of Theorem 8-2a makes use of the Least Upper Bound Principle and the following simple lemma:

Lemma A7-2a. If $\lim_{x \to a} f(x) = L$ and $L > 0$, then there is a positive number δ such that

$$f(x) > \frac{L}{2} > 0$$

if x is in the domain of f and

$$0 < |x - a| < \delta.$$

Proof. The definition of limit tells us that for any given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

if x is in the domain of f and

$$0 < |x - a| < \delta.$$

By assumption $L > 0$, so that $\frac{L}{2}$ is also positive. Therefore, (taking $\epsilon = \frac{L}{2}$) we can find a positive number δ so that

(1) $$|f(x) - L| < \frac{L}{2}$$

if x is in the domain of f and

$$0 < |x - a| < \delta.$$

The inequality (1) can be rewritten as

$$-\frac{L}{2} < f(x) - L < \frac{L}{2}.$$

Adding L to both sides we have

$$-\frac{L}{2} + L < f(x) < \frac{L}{2} + L$$

so that, in particular, $f(x)$ cannot be less than $L - \frac{L}{2} = \frac{L}{2} > 0$. This completes the proof of the lemma.

This lemma has been implicitly used before in the form of the assertion that if $f(x)$ approximates a positive number as x approaches a, then the values $f(x)$ must be positive if x is close enough to a.

Proof of Theorem 8-2a. We give the proof for the case when $f(a) < f(b)$. The proof for the case $f(a) > f(b)$ is analogous. Suppose that $f(a) < d < f(b)$. Our purpose is to show that there is a number c such that $a < c < b$ and $f(c) = d$. Such a number can be found as follows: Let A be the set of all numbers x in the interval $[a,b]$ such that $f(x) \leq d$.

The set A is certainly not empty (since $a \in A$) and bounded above (by b). The Least Upper Bound Principle implies the existence of a number c such that

(2) $$x \leq c \quad \text{if} \quad x \in A$$

and

(3) $$c \leq \alpha \quad \text{if} \quad \alpha \text{ is any upper bound for } A.$$

We shall show that $a < c < b$ and that $f(c) = d$. First we note that $a \leq c$ (since (2) holds and $a \in A$) and that $c \leq b$ (since (3) holds and b is an upper bound for A). To show that $a < c$, consider the function g defined by

(4) $$g(x) = d - f(x), \quad a \leq x \leq b.$$

By assumption $d > f(a)$, so that $g(a) > 0$. Furthermore,

$$\begin{aligned}\lim_{x \to a} g(x) &= \lim_{x \to a} d - \lim_{x \to a} f(x) \\ &= d - f(a)\end{aligned}$$

since f is continuous at a. Therefore,

$$\lim_{x \to a} g(x) > 0$$

and we can apply Lemma A7-2a to conclude that if x is close enough to a and $x \geq a$ then

$$g(x) > 0.$$

In particular, there is an x in $[a,b]$ such that $x > a$ and $g(x) > 0$, that is,

$$f(x) < d.$$

Such an x must belong to A (from the definition of A) so that (2) implies $c \geq x > a$. A similar argument (applied to $h(x) = f(x) - d$, instead of g) shows that $c < b$. This completes the proof that $a < c < b$.

Now we show that $f(c) = d$. Suppose this is false, so that $f(c) < d$ or $f(c) > d$. Consider the case $f(c) < d$, and again let

$$g(x) = d - f(x).$$

Since f is continuous at c, we have

$$\begin{aligned} \lim_{x \to c} g(x) &= \lim_{x \to c} d - \lim_{x \to c} f(x) \\ &= d - f(c) > 0. \end{aligned}$$

Again apply Lemma A7-2a to conclude that

$$g(x) > 0$$

if x is sufficiently close to c, and $g(x)$ is defined. Since $g(x)$ is defined for $c \leq x \leq b$ and $b > c$, there must be a point x, such that $c < x \leq b$ and $g(x) > 0$, that is $f(x) < d$. Such an x must belong to A so that $x \leq c$ (from (2)). This contradicts the fact that $c < x \leq b$.

The assumption that $f(c) < d$ has led us to a contradiction. A similar argument (applied to $h(x) = f(x) - d$, instead of g) shows that the assumption $f(c) > d$ must also lead to a contradiction. We are forced to conclude that indeed $f(c) = d$. This completes the proof of Theorem 8-2a.

Our proof of Theorem 8-2b will make use of the following lemma whose proof is a simple consequence of the definition of limit.

Lemma A7-2b. If $\lim_{x \to a} f(x) = f(a)$, then there is a number $\delta > 0$ such that

$$|f(x)| \leq 1 + |f(a)|$$

for all x in the domain of f such that

$$a - \delta < x < a + \delta.$$

Proof. Since $\lim_{x \to a} f(x) = f(a)$, the definition of limit tells us that for any given $\epsilon > 0$ we can find a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon$$

if x is in the domain of f and

$$0 < |x - a| < \delta.$$

In particular, we can find a positive number δ such that

(5) $$|f(x) - f(a)| < 1$$

if x is in the domain of f and

(6) $$0 < |x - a| < \delta.$$

The inequality (5) certainly holds if $x = a$, (for then $|f(x) - f(a)| = |f(a) - f(a)| = 0$) so (6) can be replaced by

(7) $$0 \leq |x - a| < \delta.$$

If x is in the domain of f then

$$f(x) = f(x) - f(a) + f(a)$$

so that the triangle inequality gives

$$|f(x)| \leq |f(x) - f(a)| + |f(a)|.$$

Thus, if x also satisfies (7) we can apply (5) to conclude that

$$|f(x)| \leq 1 + |f(a)|.$$

This is our desired result for (7) and can be rewritten as

$$a - \delta < x < a + \delta.$$

Proof of Theorem 8-2b. Suppose f is continuous at each point of the interval $a \leq x \leq b$. We first show that f is bounded on the interval, that is,

(8) there is a number M such that $|f(x)| \leq M$ for $a \leq x \leq b$.

Let A be the set of numbers t, such that

(9) $a \leq t \leq b$ and f is bounded on the interval $a \leq x \leq t$.

Certainly A is not empty (for $a \in A$) and bounded above by b, so it has a least upper bound, say α. We shall show that $\alpha \in A$ and that $\alpha = b$. This will establish that $b \in A$ and hence that (8) holds.

The number α, being the least upper bound of A, satisfies the two conditions

(10) $t \leq \alpha$ if $t \in A$ (α is an upper bound for A)

and

(11) if $t \leq \beta$ for all $t \in A$ then $\alpha \leq \beta$, (α is not larger than any other upper bound β).

Since $a \in A$, it follows from (10) that $a \leq \alpha$. Also since b is an upper bound for A, it follows from (11) that $\alpha \leq b$. Therefore, f must be continuous at α, so that

$$\lim_{x \to \alpha} f(x) = f(\alpha).$$

Apply A7-2b to conclude that there is a positive number δ such that

(12) $$|f(x)| \leq 1 + |f(\alpha)|$$

if x is in the domain of f and

(13) $$\alpha - \delta < x < \alpha + \delta.$$

This will be used to show that $\alpha \in A$ and that $\alpha = b$.

To show that $\alpha \in A$, we first observe that $\alpha - \delta < \alpha$ so that $\alpha - \delta$ cannot be an upper bound for A (from (11)). Therefore, there is at least one $t \in A$ such that $t > \alpha - \delta$. Such a number t cannot exceed α (from (10)). Furthermore, the values of f must be bounded in the interval $a \leq x \leq t$ (from (9)), so there is a number M_1 such that

(14) $$|f(x)| \leq M_1, \quad a \leq x \leq t.$$

Noting that if $t \leq x \leq \alpha$, then

$$\alpha - \delta < x < \alpha + \delta$$

(since $\alpha - \delta < t \leq x \leq \alpha < \alpha + \delta$) we conclude from (12) that

(15) $$|f(x)| \leq 1 + |f(\alpha)|, \text{ if } t \leq x \leq \alpha.$$

Let M_2 be the larger of M_1 and $1 + |f(x)|$ then (14) and (15) tell us that

(16) $$|f(x)| \leq M_2, \quad a \leq x \leq \alpha,$$

so that f is bounded on the interval $[a,\alpha]$ and hence α must be in A.

To show that $\alpha = b$, we first recall that $\alpha \leq b$ (from (11)). If it were true that $\alpha < b$, then since $\alpha < \alpha + \delta$, we can find a number t_1 in the interval $[a,b]$ such that

(17) $$\alpha < t_1 < \alpha + \delta.$$

Therefore, if $\alpha \leq x \leq t_1$ then (12) gives

(18) $$|f(x)| \leq 1 + |f(\alpha)|, \text{ if } \alpha \leq x \leq t_1$$

(since $\alpha - \delta < \alpha \leq x \leq t_1 < \alpha + \delta$, so that (13) holds). Let M_3 be the larger of M_2 (of (16)) and $1 + |f(\alpha)|$. Combining (16) and (18) we have

$$|f(x)| \leq M_3 \text{ if } a \leq x \leq t_1$$

so that t_1 must belong to A, and, hence, $t_1 \leq \alpha$. This contradicts (17) and we are forced to conclude that α cannot be less than b. This completes the proof of (8).

We now complete the proof of Theorem 8-2b. Let B be the image of the interval $[a,b]$ under f, that is,

(19) $$B \text{ is the set of all numbers } f(x), \quad a \leq x \leq b.$$

The set B is non-empty (since $f(a) \in B$) and bounded above (from (8)) so it has a least upper bound, which we denote by α. Thus

(20) $$f(x) \leq \alpha \text{ if } a \leq x \leq b \quad (\alpha \text{ is an upper bound for } B)$$

and

(21) $$\text{if } f(x) \leq \beta \text{ for } a \leq x \leq b \text{ then } \alpha \leq \beta$$
(α is not larger than any other upper bound for B).

It will be shown that there is a number c in $[a,b]$ such that $f(c) = \alpha$. From (20) we will then have

$$f(x) \leq f(c) \quad \text{for} \quad a \leq x \leq b$$

so that $f(c)$ is our desired maximum value of f on the interval $[a,b]$.

Suppose there is no c in $[a,b]$ such that $f(c) = \alpha$. Since, $f(x) \leq \alpha$, $a \leq x \leq b$, we must, therefore, have $f(x) < \alpha$, $a \leq x \leq b$, so that the function g defined by

$$g(x) = \frac{1}{\alpha - f(x)}$$

is defined for each x in $[a,b]$ (for the denominator is not zero in the interval). Furthermore, for each t in $[a,b]$, we then have

$$\lim_{x \to t} g(x) = \frac{1}{\lim_{x \to t} (\alpha - f(x))} = \frac{1}{\alpha - \lim_{x \to t} f(x)}$$

$$= \frac{1}{\alpha - f(t)} = g(t)$$

so that g is continuous at each point of $[a,b]$. Apply (8) to g to conclude that there is an M such that

$$|g(x)| \leq M, \quad a \leq x \leq b.$$

For each x in $[a,b]$, we have:

$$g(x) = \frac{1}{\alpha - f(x)} > 0$$

so that

$$0 < \frac{1}{\alpha - f(x)} \leq M.$$

Taking reciprocals we have

$$\alpha - f(x) \geq \frac{1}{M}$$

that is;

$$f(x) \leq \alpha - \frac{1}{M} \quad \text{for} \quad a \leq x \leq b.$$

Hence, $\alpha - \frac{1}{M}$ is an upper bound for B. This contradicts (21) since $\alpha > \alpha - \frac{1}{M}$. This contradiction was a consequence of the assumption that there is no c in $[a,b]$ such that $f(c) = \alpha$. Hence, there must be such a c, that is, there is a number c in $[a,b]$ such that

$$f(x) \leq f(c), \quad a \leq x \leq b.$$

The proof that there is a d in $[a,b]$ such that $f(d) \leq f(x)$, $a \leq x \leq b$ is analogous. Of course, now that we know that continuous functions on $[a,b]$ have maximums, we can apply this to the function $-f$. A maximum for $-f$ will be a minimum for f so that continuous functions on closed intervals must have both maximum and minimum points.

Exercises A7-2

1. Let

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

Show that f satisfies the conclusion of Theorem 8-2a on any interval $[0,b]$, but f is not continuous at $x = 0$.

2. Prove that if f is continuous and has an inverse on $[a,b]$ and $f(a) < f(b)$ then f is strictly increasing.

3. Prove that if f is continuous on $[a,b]$ then the image of $[a,b]$ is a closed interval. (Hint: Use Theorems 8-2a and b).

4. Prove that if f is continuous in $[a,b]$ and all values of f are in $[a,b]$ then there is an x in $[a,b]$ for which $f(x) = x$.

5. Suppose

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Does f satisfy the hypothesis of Theorem 8-2b on the interval $[0,1]$? Does (8) hold for f on $[0,1]$? on $[10^{-100},1]$?

6. Is the continuity of f essential to the hypothesis of (8)?

7. Can a discontinuous function whose domain is a closed interval be bounded?

8. Do Numbers 6 and 7 amount to the same question?

9. Can a nonconstant function whose domain is the set of real numbers be bounded?

10. Show that a function f which is increasing in a neighborhood at each point of an interval $[a,b]$ is an increasing function in $[a,b]$. (Hint: Let A be the set of all t, in $[a,b]$ such that f is increasing in $[a,t]$. Show that if $\alpha = \text{lub } A$, then $\alpha \in A$ and $\alpha = b$).

11. A function has the property that for each point of an interval where it is defined, there is a neighborhood in which the function is bounded. Show that the function is bounded over the whole interval. (This is an example as is Number 10 where a local property implies a global one. It is clear that the global property here implies the local one.)

A7-3. The Mean Value Theorem

In Section 8-3 we discussed the Mean Value Theorem. We amplify that discussion here.

The Mean Value Theorem.

Suppose f is continuous at each point of the interval $a \leq x \leq b$ and differentiable at each point of $a < x < b$. Then there is at least one number c, such that $a < c < b$ and

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

In this section we give a proof of this result, and show how it can be used to obtain error estimates in approximation formulas. Further applications will be discussed in the next section.

In geometrical terms, the Mean Value Theorem states that on the arc between any two points of the graph of a differentiable function there exists a point where the curve has the same slope as the chord.* Thus, let $(p,f(p))$ and $(q,f(q))$ be any two points on the graph of a differentiable function f with $p < q$, say (see Figure A7-3a).

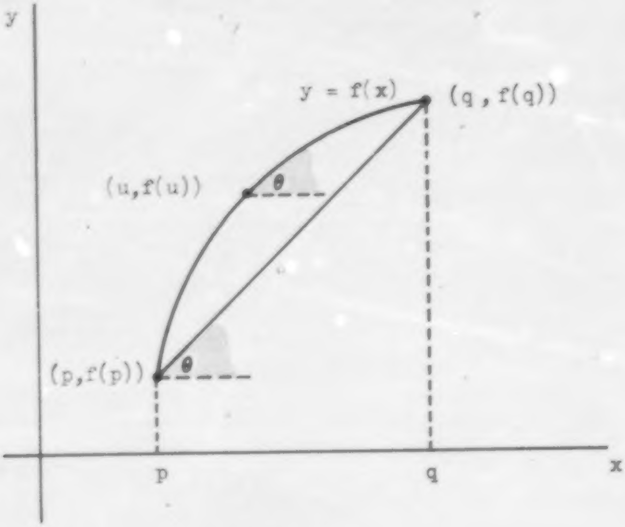

Figure A7-3a

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[*] The word "mean" here signifies "average". The slope of the chord is interpreted as average rise in function value per rise in value of x. The Mean Value Theorem states that this average is equal to a value of the derivative at some point of the interval.

According to the Mean Value Theorem there exists a point u between p and q where

$$f'(u) = \frac{f(q) - f(p)}{q - p} .$$

We can make the Mean Value Theorem plausible by an argument similar to that by which we found that the slope of a graph at an interior extremum is zero. Take a parallel to the chord at a point $(u,f(u))$ which lies on the arc at maximum distance from the chord. Since no point of the arc lies at a greater distance from the chord, the arc cannot cross the parallel. The arc cannot meet the parallel at an angle for then it would cross; therefore the two must have the same direction at $(u,f(u))$. (See Figure A7-3b.)

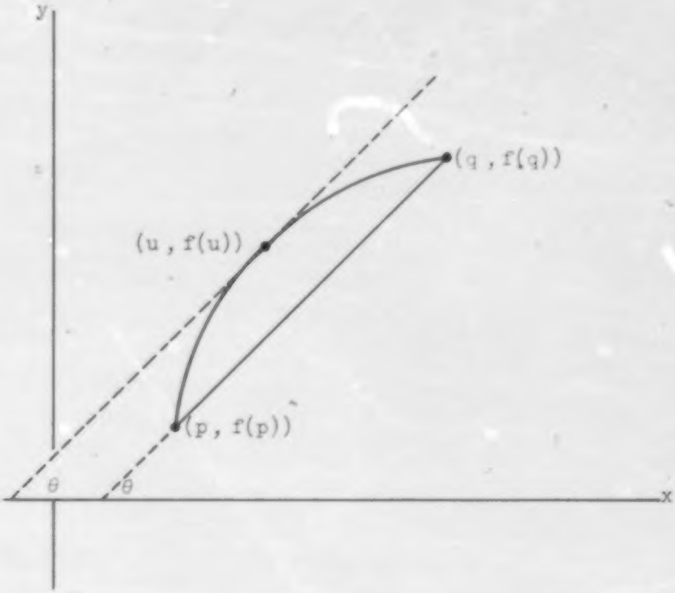

Figure A7-3b

In order to derive the Mean Value Theorem we first prove it for the special case in which the chord is horizontal.

Lemma A7-3 (Rolle's Theorem). If f is continuous in the closed interval $[a,b]$, differentiable in the open interval (a,b) and $f(a) = f(b)$ then there is at least one c in (a,b) such that $f'(c) = 0$.

Proof. If f is constant then this is certainly true for any c in (a,b). If f is not constant, then there is a point α in (a,b) such that $f(\alpha) \neq f(a)$. Let us suppose $f(\alpha) > f(a)$ (otherwise we can apply the same arguments to $-f$), so that if c is a maximum point for f (which exists by Theorem 8-2b) then $f(c) > f(a)$. Certainly c must be in (a,b) (for $f(a) = f(b)$) and, hence, Theorem 8-2c implies that $f'(c) = 0$.

Before proving the Mean Value Theorem let us examine some of the other consequences of Rolle's Theorem (Lemma A7-3).

Corollary 1. Let f be differentiable on an interval. Any zeros of f within the interval are separated by zeros of the derivative.

Proof. If $x_1 < x_2$ and $f(x_1) = f(x_2) = 0$, the conditions of Lemma A7-3 are satisfied and there exists a value u such that $x_1 < u < x_2$ and $f'(u) = 0$.

As a consequence of this result we observe further that, in a given interval, a function may have at most one more zero than its derivative. From this fact there follows a familiar result:

Corollary 2. A polynomial of degree n can have no more than n distinct real zeros.

The proof is left as an exercise (Exercises A7-3, No. 1).

Example A7-3.

(i) Let us apply Corollary 1 to the zeros of $f(x) = x^3 - 3x + 1$. We know that $f'(x) = 3x^2 - 3$ has zeros at $x = 1$ and $x = -1$. It follows that f may have as many as three zeros. We observe that $f(-1) = 3$ and $f(1) = -1$. By the Intermediate Value Theorem we conclude that there is a zero of f between -1 and 1. Clearly we can make $f(x)$ negative for sufficiently large negative values and positive for sufficiently large positive values. It follows that f has a zero for $x < -1$ and another for $x > 1$. Specifically, we have $f(-2) = -1$ and $f(2) = 3$, so that there is one zero between -2 and -1 and another between 1 and 2.

(ii) The function $f(x) = x^3 + 3x + 1$ has the derivative $f'(x) = 3x^2 + 3$ which is always positive. Since the derivative is always positive f can have at most one zero. Observing that $f(-1) = -3$ and $f(0) = 1$ we see that a zero exists and lies between $x = -1$ and $x = 0$.

Proof of Theorem 8-2g. The equation of the straight line joining the points $(a,f(a))$ and $(b,f(b))$ is

$$y = g(x) = f(a) + (x - a)\,\frac{f(b) - f(a)}{b - a}\,. \tag{2}$$

It follows for any point x in (a,b) that the height $h(x)$ of $(x,f(x))$ above the chord is given by

$$h(x) = f(x) - g(x) = f(x) - f(a) - (x - a)\,\frac{f(b) - f(a)}{b - a}\,. \tag{3}$$

From this equation it follows straightforwardly that $h(x)$ satisfies the conditions of Rolle's Theorem (Lemma A7-3) on $[a,b]$. First, as you may verify directly, $h(a) = h(b) = 0$. Next observe that $h(x) = f(x) - g(x)$ is the sum of $f(x)$ and a linear function; since both terms of this sum are differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$ it follows that h also is differentiable on the open interval and continuous on the closed interval. From Rolle's Theorem, we conclude that for some value in (a,b)

$$h'(c) = f'(c) - g'(c) = 0,$$

or, from Equation (3) for $h(x)$ above,

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

Linear Interpolation

Linear interpolation is a useful method of approximation to the values of a function in an interval when the endpoint values are known. If bounds on the range of the derivative can be obtained, the Mean Value Theorem gives a way of estimating the error of approximation.

Geometrically, linear interpolation consists of replacing the arc of the graph of f on (a,b) by the chord joining the endpoints. Thus, on (a,b) we approximate $f(x)$ by the linear function $g(x)$ given in Equation (2). The error of the approximation $g(x) - f(x) = -h(x)$ is given by Equation (3). For our purposes it is convenient to recast Equation (3) in the form

$$g(x) - f(x) = (x - a)\left(\frac{f(a) - f(b)}{b - a} - \frac{f(x) - f(a)}{x - a}\right).$$

Now, by the Mean Value Theorem

$$g(x) - f(x) = (x - a)[f'(u_2) - f'(u_1)] \tag{4}$$

where $a < u_1 < x < b$, $a < u_2 < b$. If the derivative is bounded in (a,b), say $|f'(z)| \leq M_1$ for z in (a,b), then from Equation (4)

$$|g(x) - f(x)| \leq |x - a|(|f'(u_2)| + |f'(u_1)|)$$

whence

$$|g(x) - f(x)| \leq 2M_1|x - a|. \tag{5}$$

Example A7-3b. Let us estimate $\sqrt{10}$ by linear interpolation for the function $f : x \to \sqrt{x}$. Since $3 < \sqrt{10} < 4$ we take $a = 9$ and $b = 16$ in Equation (2) and obtain $g(10) = \frac{22}{7}$ as our estimate for $\sqrt{10}$. On the interval $(9,16)$ we have

$$f'(x) = \frac{1}{2\sqrt{x}} < \frac{1}{2\sqrt{9}} \leq \frac{1}{6}.$$

Entering this bound in (5) we obtain

$$\left|\frac{22}{7} - \sqrt{10}\right| \leq \frac{1}{3}.$$

We observe, however, that

$$\left(\frac{22}{7}\right)^2 = \frac{484}{49} = 10 - \frac{6}{49}$$

and we suspect that our estimate of error is rather crude.

If on the interval (a,b) f' has a derivative f'', the second derivative of f, we may apply the Mean Value Theorem again to the difference $f'(u_2) - f'(u_1)$ in Equation (4) to obtain

$$g(x) - f(x) = (x - a)(u_2 - u_1)f''(v)$$

where v is somewhere between u_2 and u_1. Since u_2 and u_1 are both points of (a,b) we know that the distance between the two points is less than the length of the interval:

$$|u_2 - u_1| < b - a.$$

Suppose, in addition, that we have a bound on the second derivative, $|f''(x)| \leq M_2$ on (a,b). Then we obtain an upper estimate for the error in terms of the second derivative:

$$|g(x) - f(x)| \leq (x - a)(b - a)M_2. \tag{6}$$

Example A7-3c. Now let us use Formula (6) to obtain an estimate for the error of approximation to $\sqrt{10}$ by the linear interpolation scheme of Example A7-3b. We have

$$|f''(x)| = \left|-\frac{1}{4x^{3/2}}\right| < \left|\frac{1}{4.9^{3/2}}\right|$$
$$\leq \frac{1}{108}$$

for x in $(9,16)$. Consequently, from (6),

$$\left|\frac{22}{7} - \sqrt{10}\right| \leq \frac{7}{108} < .065.$$

It follows that

$$3.07 < \sqrt{10} < 3.21.$$

We have obtained sharper estimates for $\sqrt{10}$ and now we can repeat the process to obtain still sharper estimates using $a = (3.07)^2$ and $b = (3.21)^2$.

Exercises A7-3

1. Prove Corollary 2 to Lemma A7-3.

2. Sketch the graphs of the functions in Example A7-3a.

3. Is the following converse of Rolle's Theorem true? If f is continuous on the closed interval $[p,q]$ and differentiable on the open interval (p,q), and if there is at least one point u in the open interval where $f'(u) = 0$, then there are two points m and n where $p \leq m < u < n \leq q$ such that $f(m) = f(n)$.

4. Does Rolle's Theorem justify the conclusion that $\frac{dy}{dx} = 0$ for some value of x in the interval $-1 \leq x \leq 1$ for $(y + 1)^3 = x^2$?

5. Given: $f(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$. Determine how many solutions $f'(x) = 0$ has and find intervals including each of these without calculating $f'(x)$.

6. Verify that Rolle's Theorem (Lemma A7-3) holds for the given function in the given interval or give a reason why it does not.

   (a) $f : x \rightarrow x^3 + 4x^2 - 7x - 10,$ $[-1,2]$

   (b) $f : x \rightarrow \frac{2 - x^2}{x^4},$ $[-1,1]$

7. Prove that the equation

$$f(x) = x^n + px + q = 0$$

cannot have more than two real solutions for an even integer n nor more than three real solutions for an odd n. Use Rolle's Theorem.

8. A function g has a continuous second derivative on the closed interval $[a,b]$. The equation $g(x) = 0$ has three different solutions in the open interval (a,b). Show that the equation $g''(x) = 0$ has at least one solution in the open interval (a,b).

9. Show that the conclusion of the Mean Value Theorem does not follow for $f(x) = \tan x$ in the interval $1.5 < x < 1.6$.

10. For each of the following functions show that the Mean Value Theorem fails to hold on the interval $[-a,a]$ if $a > 0$. Explain why the theorem fails.

    (a) $f : x \to |x|$

    (b) $f : x \to \frac{1}{x}$

11. Show that the equation $x^5 + x^3 - x - 2 = 0$ has exactly one solution in the open interval $(1,2)$.

12. Show that $x^2 = x \sin x + \cos x$ for exactly two real values of x.

13. Find a number that can be chosen as the number C in the Mean Value Theorem for the given function and interval.

    (a) $f : x \to \cos x, \quad 0 \leq x \leq \frac{\pi}{2}$

    (b) $f : x \to x^3, \quad -1 \leq x \leq 1$

    (c) $f : x \to x^3 - 2x^2 + 1, \quad -1 \leq x \leq 0$

    (d) $f : x \to \cos x + \sin x, \quad 0 \leq x \leq 2\pi$

14. Derive each of the following inequalities by applying the Mean Value Theorem.

    (a) $|\sin x - \sin y| \leq |x - y|$

    (b) $\dfrac{x}{1 + x^2} < \arctan x < x$ if $x > 0$

15. Use the Mean Value Theorem to approximate $\sqrt[3]{1.008}$.

16. Use the Mean Value Theorem to approximate $\cos 61°$.

17. Show that $a\left(1 + \dfrac{\epsilon}{n(a^n + \epsilon)}\right) < \sqrt[n]{a^n + \epsilon} < a\left(1 + \dfrac{\epsilon}{na^n}\right)$

    for $\epsilon > 0$, $a > 1$, $n > 1$ (n rational).

18. Using Number 17, obtain the following approximations.

    (a) $3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9}$

    (b) $3 + \frac{3}{5(244)} < \sqrt[5]{244} < 3 + \frac{1}{405}$

    (c) Show that the approximation

    $$\frac{1}{2}\left(3 + \frac{3}{5(244)} + 3 + \frac{1}{405}\right) \text{ to } \sqrt[5]{244}$$

    is correct to at least 5 decimal places.

19. (a) Show that a straight line can intersect the graph of a polynomial of n-th degree at most n times.

(b) Obtain the corresponding result for rational functions.

(c) Could $\sin x$ or $\cos x$ be rational functions? Justify your answer.

20. Prove the intermediate value property for derivatives; namely, if f is differentiable on the closed interval $[p,q]$ then $f'(x)$ takes on every value between $f'(p)$ and $f'(q)$ in the open interval (p,q).

21. Estimate for Newton's Method. (See Section 2- .) Suppose f' and f'' are positive on $[a,b]$ and that $f(r) = 0$, where $r \in [a,b]$. Let $x_1 \in [a,b]$ and put

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} .$$

Suppose

$$|f''(x)| < M \text{ and } |f'(x)| \geq m > 0, \quad a \leq x \leq b.$$

(a) Show that

$$|x_2 - r| \leq |x_1 - r|^2 \frac{M}{m} .$$

(Hint: $x_2 - r = x_1 - r - \dfrac{f(x_1) - f(r)}{f'(x_1)}$. Find ξ between x_1 and r such that

$$x_2 - r = x_1 - r - \frac{f'(\xi)}{f'(x_1)}(x_1 - r)$$
$$= \frac{f'(x_1) - f'(\xi)}{f'(x_1)}(x_1 - r).$$

Then find ξ_1 between x_1 and ξ

$$x_2 - r = \frac{f''(\xi_1)}{f'(\xi_1)}(x_1 - r)(x_1 - \xi).$$

(b) If $b - a < \frac{m}{M} k$, $0 < k < 1$, show that $|x_2 - r| \leq \frac{m}{M} k^2$.

(c) Prove (0) of Section 2- .

A7-4. Applications of The Mean Value Theorem

This is an extension of some of the ideas of Section 8-4.

THEOREM A7-4a. If $f'(x) \geq 0$ for $a < x < b$, then f is increasing on (a,b); if $f'(x) \leq 0$ then f is decreasing.

Proof. Only the increasing case $f'(x) \geq 0$ will be considered here, the case $f'(x) \leq 0$ is similar (or can be obtained by considering $-f$).

Suppose $f'(x) \geq 0$ on (a,b). For any two numbers x_1 and x_2 in the interval with $x_1 < x_2$, the Mean Value Theorem tells us that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c in (x_1,x_2). Since $f'(c) \geq 0$ we must have

$$f(x_2) - f(x_1) \geq 0, \text{ that is } f(x_1) \leq f(x_2).$$

This proves the theorem.

If we replace the weak inequalities $(>$, and $\leq)$ by the stronger inequalities $(>$, and $<$, respectively) the same proof yields

THEOREM A7-4b. If $f'(x) > 0$ for $a < x < b$ then f is strictly increasing in (a,b); if $f'(x) < 0$ then f is strictly decreasing.

Theorem 7-3b is a simple corollary to Theorem A7-4b, for if

$$F'(x) = G'(x) \quad \text{for } a \leq x \leq b$$

then $(F - G)' = 0$ on $a < x < b$ so that $F - G$ is both increasing and decreasing and, hence, must be constant on (a,b), that is

$$F(x) = G(x) + c, \quad a < x < b,$$

where c is a constant. This also holds at the endpoints a and b, since F and G must also be continuous at a and b.

THEOREM A7-4c. If $f''(x) \geq 0$ for $a < x < b$, then f is convex on (a,b); if $f''(x) \leq 0$ then f is concave.

Proof. We discuss only the case $f''(x) \geq 0$. We wish to show that if x_1 and x_2 are in (a,b) and $x_1 < x_2$, then

$$f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1) \tag{1}$$

(See (3) of Section 8-2). To prove this, we apply the Mean Value Theorem to find a c in (x_1,x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Then

$$\begin{aligned} f(x_2) - f(x_1) - f'(x_1)(x_2 - x_1) &= f'(c)(x_2 - x_1) - f'(x_1)(x_2 - x_1) \\ &= (f'(c) - f'(x_1))(x_2 - x_1). \end{aligned}$$

Apply the Mean Value Theorem to $f'(c) - f'(x_1)$. This gives a number c_1 in (x_1,c) such that

$$f'(c) - f'(x_1) = f''(c_1)(c - x_1).$$

Hence

$$f(x_2) - f(x_1) - f'(x_1)(x_2 - x_1) = f''(c_1)(c - x_1)(x_2 - x_1)$$

which is non-negative, since $f''(c_1) \geq 0$ and $c > x_1$, $x_2 > x_1$. Therefore,

$$f(x_2) - f(x_1) - f'(x_1)(x_2 - x_1) \geq 0$$

which is the same as (1). A similar argument shows that (1) holds if $x_2 < x_1$.

Exercises A7-4

1. Let f be differentiable on a neighborhood of a point a for which $f'(a) = 0$. If $f'(x) \leq 0$ when $x < a$ and $f'(x) \geq 0$ when $x > a$, then $f(a)$ is a minimum. If $f'(x) \geq 0$ when $x < a$ and $f'(x) \leq 0$ when $x > a$ then $f(a)$ is a maximum. Give a proof.

2. Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Suppose u is the one point in (a,b) where $f'(u) = 0$. Prove that if $f'(x)$ reverses sign in a neighborhood of u then $f(u)$ is the global extremum of f on $[a,b]$ appropriate to the sense of reversal.

3. Given a function f such that $f(1) = f(2) = 4$, and such that $f''(x)$ exists and is positive throughtout the interval $1 \leq x \leq 3$.

   (a) What can you conclude about $f'(2.5)$?

   (b) Prove your statement, stating whatever theorems you use in your proof.

4. Let f be a differentiable function on (a,b). Prove that the requirement that f be increasing is equivalent to the condition that $f'(x) \geq 0$ everywhere but that every interval contains points where $f'(x) > 0$.

5. A function g is such that g'' is continuous and positive in the interval (p,q). What is the maximum number of roots of each of the equations $g(x) = 0$ and $g'(x) = 0$ in (p,q)?
   Prove your result and give some illustrative examples.

6. Suppose that $f^{(1)}(a) = f^{(2)}(a) = \ldots = f^{(n-1)}(a) = 0$ but that $f^{(n)}(a) \neq 0$. Determine whether $f(a)$ is a local extremum, and if it is, which kind. (Hint: Consider separately the cases n even and n odd.)

7. Prove that a necessary and sufficient condition that the graph of a differentiable function f be concave on an interval I is that for each point a in I, the slope of the chord joining a point $(x,f(x))$ to the fixed point $(a,f(a))$ is a decreasing function of x on I.

8. (a) Let f be differentiable and its graph is concave on an interval I. Prove that the function

$$\phi(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

is decreasing, where the fixed point a is any interior point of I.

(b) From the result of (a), prove that a necessary and sufficient condition that the graph of f be concave on I is that f' be decreasing.

9. (a) Let x and y be two points on an interval I in the domain of a function f. Show that a point is on the chord joining the points $(x,f(x))$ and $(y,f(y))$ on the graph of f if, and only if, its coordinates are

$$(\theta x + (1 - \theta)y, \theta f(x) + (1 - \theta)f(y))$$

for some θ such that $0 \leq \theta \leq 1$.

(b) Show that a differentiable function f is convex on I if, and only if, for all x and y in I and all θ such that $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$

(c) Use (b) to show that the graphs of the following functions are convex.

(i) $f : x \to ax + b$

(ii) $f : x \to x^2$

(iii) $f : x \to -\sqrt{x}$

10. (a) Derive the following property of differentiable functions. If the graph of f is concave on an interval I, then for all points a, b in I and any positive numbers p, q

$$f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q}.$$

In words, the function value of a weighted average is not less than the weighted average of the function values.

(b) Prove that this property is sufficient for concavity.

11. Prove that if f is differentiable then a necessary and sufficient condition for its graph to be concave is that

$$f(\frac{a+b}{2}) \geq \frac{f(a) + f(b)}{2} .$$

12. The graph of a differentiable function f is concave and is positive for all x. Show that f is a constant function.

13. Under what circumstances will the graph of a function f and its inverse both be concave? one concave and the other convex?

14. If either of $D^2xF(x)$ or $D^2F(\frac{1}{x})$ is of one sign for $x > 0$, show that the other one has the same sign. Interpret geometrically and illustrate by several examples.

15. If $F(x)$ is concave and $F(a) = F(b) = F(c)$ where $a < b < c$, show that $F(x)$ is constant in (a,c).

16. (a) Let a, b, and c be points in I such that $a < b < c$, and suppose that the graph of f is convex in I. Show that

$$f(b) \leq \frac{c - b}{c - a} f(a) + \frac{b - a}{c - a} f(c).$$

(Hint: Use the result of Number 13.); hence,

$$f(a) \geq \frac{c - a}{c - b} f(b) - \frac{b - a}{c - b} f(c),$$

$$f(c) \geq \frac{c - a}{b - a} f(b) - \frac{c - b}{b - a} f(a).$$

(b) If the graph of F is convex in a closed interval, show that F is bounded in the interval.

(c) Show by a counter-example that the result in (b) is not valid for an open interval.

Appendix 8

MORE ABOUT INTEGRALS

A8-1. Existence of the Integral

The purpose of this section is to establish necessary and sufficient conditions for the existence of the integral of a function f over $[a,b]$. Recall that the integral is defined as the unique separation number between the upper and lower sums. We need first to establish that the upper and lower sums are in fact separated: that every lower sum is less than or equal to every upper sum. If it is possible to find an upper sum and a lower sum closer together than any given fixed tolerance ϵ, than by Lemma A1-5 there exists a unique separation number, a number I which is the integral of f over $[a,b]$.

Lemma A8-1a. Let f be a function defined and bounded on $[a,b]$. For any fixed partition σ of $[a,b]$, each upper sum U over σ is greater than or equal to each lower sum L over σ.

Proof. We recall that the partition σ is simply a set of points of $[a,b]$ which includes the endpoints a and b. To construct upper and lower sums, the points of σ are arranged in increasing order; i.e.,

$$a = x_0 \le x_1 \le x_2 \le \dots \le x_n = b.$$

An upper sum U is defined as

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

where $f(x) \le M_k$ on $[x_{k-1},x_k]$, a lower sum as

$$L = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

where $f(x) \ge m_k$ on $[x_{k-1},x_k]$. Thus $m_k \le M_k$ and term-for-term

$$m_k(x_k - x_{k-1}) \le M_k(x_k - x_{k-1})$$

from which this lemma follows.

It is necessary to find a means of comparing upper and lower sums for any two partitions σ_1 and σ_2. For this purpose we introduce the joint partition $\sigma = \sigma_1 \cup \sigma_2$ which consists of all points of the two partitions taken together. Let U_1 be any upper sum over σ_1 and L_2 any lower sum over σ_2. We shall show that U_1 is an upper sum for the joint subdivision σ and that L_2, similarly, is a lower sum for σ. The result we seek will then follow from the preceding lemma.

Lemma A8-1b. For any partitions σ_1 and σ_2 of $[a,b]$ and any upper and lower sums U_1, L_2, over the respective subdivisions,

$$U_1 \geq L_2.$$

Proof. Let x_{k-1}, x_k be a pair of consecutive points of subdivision from σ_1, $(k = 1, 2, \ldots, n)$. There may be points of the subdivision σ_2 in the open interval (x_{k-1},x_k), say, $u_1, \ldots, u_{p-1}$ with $x_{k-1} < u_1 < u_2 < \ldots < u_{p-1} < x_k$. Setting $u_0 = x_{k-1}$ and $u_p = x_k$, we see that the set $\{u_i : i = 0, \ldots, p\}$ is a partition of $[x_{k-1},x_k]$. Further since M_k and m_k are upper and lower bounds for $f(x)$ in all of $[x_{k-1},x_k]$ they are bounds for $f(x)$ in each of the subintervals $[u_{i-1},u_i]$, $i = 0, 1, 2, \ldots, p$, (see Figure A8-1). If we form the upper sum U_k^* over the partition of $[x_{k-1},x_k]$ using the upper bound M_k we have

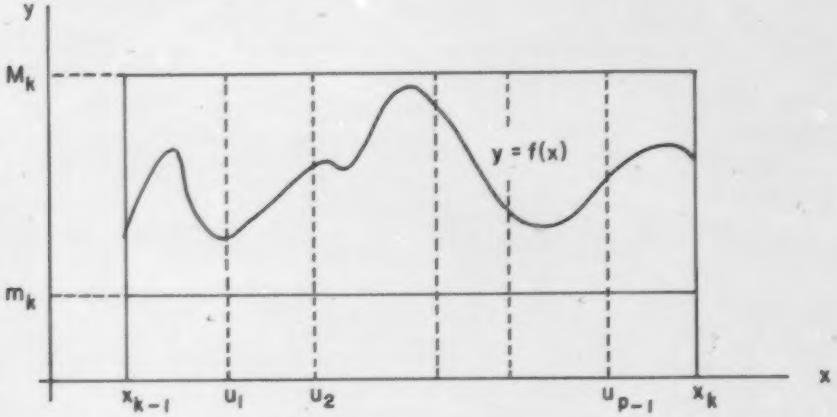

Figure A8-1

$U_k^* = \sum_{i=1}^{p} M_k(u_i - u_{i-1}) = M_k \sum_{i=1}^{p} (u_i - u_{i-1}) = M_k(x_k - x_{k-1})$. Thus the upper sum

$U_1 = \sum_{k=1}^{n} U_k^*$ for the partition σ_1 is also an upper sum for σ. Similarly L_2 is a lower sum for both σ_2 and σ. It follows from Lemma A8-1a that

$$L_2 \le U_1.$$

Corollary. If for any partitions σ_1 and σ_2 of $[a,b]$ there exist an upper sum U_1 over σ_1 and a lower sum L_2 over σ_2 satisfying

$$U_1 - L_2 < \epsilon,$$

then there exists a partition σ which has upper and lower sums U and L satisfying

$$U - L < \epsilon.$$

Proof. Take $\sigma = \sigma_1 \cup \sigma_2$. Since U_1 and L_2 are upper and lower sums for the joint partition, the result is immediate.

THEOREM 6-3a. Let f be a bounded function on $[a,b]$. If for every positive ϵ there exists a partition σ of $[a,b]$ and lower and upper sums L and U over σ which differ by less than ϵ, then there exists a number I which is the integral of f over $[a,b]$. Conversely, if f is integrable over $[a,b]$, then there exist a partition σ and lower and upper sums L and U over σ such that $U - L < \epsilon$.

Proof. From Lemma A8-2b every lower sum is less than or equal to each upper sum. If for every $\epsilon > 0$ there exist lower and upper sums L and U satisfying $U - L < \epsilon$, then by Lemma A7-1 the number separating the set of lower sums from the set of upper sums is unique. By definition this separation number is the integral of f over $[a,b]$.

Conversely, if f is integrable, that is, if the integral of f over $[a,b]$ exists, then by definition the separation number between lower and upper sums in unique. It follows from the converse statement in Lemma A7-1 that there exist lower and upper sums, not necessarily over the same partition, say L_1 over σ_1 and U_2 over σ_2 for which $U_2 - L_1 < \epsilon$. From the corollary to Lemma A8-1b, we conclude that there exists a single partition σ having upper and lower sums U and L for which $U - L < \epsilon$.

Next we prove a useful corollary to Theorem 6-2a.

Lemma A8-2c. If f is integrable over $[a,b]$ then f is integrable over any subinterval $[\alpha,\beta]$.

Proof. There exists a partition σ of $[a,b]$ for which $U - L < \epsilon$ where U and L denote upper and lower sums over σ. We may assume α and β are points of σ, for if they were not so originally they could be introduced without affecting the values of U and L (see the proof of Lemma A8-2b). With α and β included in σ, it follows that σ contains a partition σ' of $[\alpha,\beta]$. Now in the sum

$$U - L = \sum (M_k - m_k)(x_k - x_{k-1})$$

all terms are nonnegative. If we let U' and L' denote those parts of the sums U and L which are taken over σ', it follows that

$$U' - L' \leq U - L < \epsilon.$$

According to Theorem 6-3a, the function f is integrable over $[\alpha,\beta]$.

Exercises A8-1

1. Let f be a function which takes on a maximum and minimum on every closed interval (e.g., f could be a continuous function, or monotone). Let $U^*(\sigma)$ and $L^*(\sigma)$ be the upper and lower sums obtained by using the maximum and minimum values of $f(x)$ as the appropriate bounds in each interval of the subdivision.

   Let σ_1 and σ_2 be any partitions of $[a,b]$. Prove for the joint subdivision $\sigma = \sigma_1 \cup \sigma_2$ that

   $$U^*(\sigma_1) \geq U^*(\sigma) \geq L^*(\sigma) \geq L^*(\sigma_2)$$

   In other terms, by adding new points to a subdivision we may reduce the difference between the upper and lower sums, and we cannot increase it.

2. Consider the function f defined on $[0,1]$ by

   $$f(x) = \begin{cases} 0 \text{ , } x \text{ irrational} \\ 1 \text{ , } x \text{ rational} \end{cases}$$

   Prove that the integral of f does not exist.

3. Consider the function f defined on $[0,1]$ by

   $$f(x) = \begin{cases} 0 \text{ , } x \text{ irrational} \\ \frac{1}{t} \text{ , } x \text{ rational, } x = \frac{s}{t} \text{ in lowest terms.} \end{cases}$$

   Prove that the integral of f over $[0,1]$ exists and find its value.

4. Give an example of a nonintegrable function fg where f and g are each integrable.

A8.2. The Integral of a Continuous Function

In this section it will be shown that if f is continuous on the interval $[a,b]$ then the integral of f exists, that is, there is a unique separation number between the upper and lower sums.

Suppose f is continuous on $[a,b]$ and that x is a point of $[a,b]$. If σ_1 and σ_2 are any two partitions of $[a,x]$ with corresponding upper sums U_1, U_2 and lower sums L_1, L_2 then we know that

$$L_2 \leq U_1. \tag{1}$$

In particular, if A denotes the set of all possible upper sums for all possible partitions of $[a,x]$ and B denotes the set of all possible lower sums for all possible partitions of $[a,x]$ then (1) tells us that each number in B is a lower bound for A and each number in A is an upper bound for B. The symbol

$$\int_a^{\overline{x}} f \quad \text{(read "the upper integral of } f \text{ from } a \text{ to } x\text{")}$$

will denote the greatest lower bound of A. The symbol

$$\int_{\underline{a}}^{x} f \quad \text{(read "the lower integral of } f \text{ from } a \text{ to } x\text{")}$$

will denote the least upper bound of B. Since each U_1 in A is an upper-bound for B we must have

$$\int_{\underline{a}}^{x} f \leq U_1$$

so that $\int_{\underline{a}}^{x} f$ is a lower bound for A and hence cannot exceed the greatest lower bound for A, that is

$$\int_{\underline{a}}^{x} f \leq \int_a^{\overline{x}} f. \tag{2}$$

Our purpose is to show that

$$\int_{\underline{a}}^{x} f = \int_a^{\overline{x}} f \tag{3}$$

that is, there is a unique separation number for the upper and lower sums on each subinterval $[a,x]$. The method of proof is as follows: Let $\overline{F}$ and $\underline{F}$

be the functions defined by these upper and lower integrals, that is

$$\overline{F}(x) = \int_a^{\overline{x}} f, \quad a \leq x \leq b$$

$$\underline{F}(x) = \int_{\underline{a}}^{x} f, \quad a \leq x \leq b.$$

We shall show that $\overline{F}$ and $\underline{F}$ have the same derivatives (namely f) and hence their difference is constant (Theorem 7-3b). Since their values at a are the same (namely 0) they must be the same functions, which is statement (3).

Certainly $\overline{F}(a) = \underline{F}(a) = 0$. (See Exercises A5-4, No. 8), so it is enough to show that $\overline{F}' = f = \underline{F}'$. We shall establish the fact that $\overline{F}' = f$, the proof of $\underline{F}' = f$ being quite similar. In summary, we shall prove

THEOREM A8-2. If f is continuous on $[a,b]$ and $\overline{F}(x) = \int_a^{\overline{x}} f, \; a \leq x \leq b$, then

$$\overline{F}'(x) = f(x), \quad a \leq x \leq b.$$

The proof of this theorem is quite analogous to the proof of the Area Theorem (Theorem A7-3a), with some complications due to the fact that f is not assumed to be increasing. We first establish three lemmas.

Lemma A8-2a. If f is continuous on $[a,b]$ and $a < c < b$, then

$$\int_a^{\overline{b}} f = \int_a^{\overline{c}} f + \int_c^{\overline{b}} f.$$

Proof. Let σ_1 be a partition of $[a,c]$ and σ_2 a partition of $[c,b]$. The union $\sigma_1 \cup \sigma_2$ is a partition of $[a,b]$. If U_1 and U_2 devote upper sums for σ_1 and σ_2 then $U_1 + U_2$ is certainly any upper sum for $[a,b]$. The number

$$\int_a^{\overline{b}} f$$

is the greatest lower bound of the upper sums of partitions of $[a,b]$ so we must have

$$\int_a^{\overline{b}} f \leq U_1 + U_2$$

that is

$$\int_a^{\overline{b}} f - U_2 \leq U_1.$$

In other words

$$\int_a^{\overline{b}} f - U_2$$

doesn't exceed any upper sum U_1 for any partition σ_1 of $[a,c]$ and hence cannot exceed the greatest lower bound of all such upper sums for all such partitions of $[a,c]$, that is

$$\int_a^{\overline{b}} f - U_2 \leq \int_a^{\overline{c}} f.$$

This can be written as

$$\int_a^{\overline{b}} f - \int_a^{\overline{c}} f \leq U_2$$

which tells us that $\int_a^{\overline{b}} f - \int_a^{\overline{c}} f$ doesn't exceed any upper sum for any partition σ_2 of $[c,b]$ and hence cannot exceed the greatest lower bound of such sums, that is

$$\int_a^{\overline{b}} f - \int_a^{\overline{c}} f \leq \int_c^{\overline{b}} f.$$

We have, therefore, established the inequality

(4) $$\int_a^{\overline{b}} f \leq \int_a^{\overline{c}} f + \int_c^{\overline{b}} f.$$

To complete the proof of Lemma A8-2a we need to establish the reverse inequality

(5) $$\int_a^{\overline{b}} f \geq \int_a^{\overline{c}} f + \int_c^{\overline{b}} f.$$

To do this, suppose σ is a partition of $[a,b]$ with a corresponding upper sum U. It may be assumed that $c \in \sigma$, for if not we can add c to σ without disturbing the sum U. (See the proof of Lemma A8-1b). Let σ_1 be the points of σ contained in $[a,c]$ and σ_2 the points of σ contained in $[c,b]$. Let U_1 and U_2 denote the upper sums obtained from U by including only the terms of U which correspond to points of σ_1 and σ_2, respectively. Then

$$(6) \qquad U = U_1 + U_2.$$

Since U_1 is an upper sum corresponding to a partition of $[a,c]$ and $\int_a^{\bar{c}} f$ is the greatest lower bound of all upper sums of all partitions of $[a,c]$ we must have

$$\int_a^{\bar{c}} f \leq U_1.$$

Similarly, we have

$$\int_a^{\bar{b}} f \leq U_2$$

so that (6) gives

$$\int_a^{\bar{c}} f + \int_c^{\bar{b}} f \leq U.$$

In other words

$$\int_a^{\bar{c}} f + \int_c^{\bar{b}} f$$

doesn't exceed any upper sum U for any partition σ of $[a,b]$, so it cannot exceed the greatest lower bound of such sums, that is

$$\int_a^{\bar{c}} f + \int_c^{\bar{b}} f \leq \int_a^{\bar{b}} f.$$

This is the desired inequality (5), which combined with (4) completes the proof of Lemma A8-2a.

Lemma A8-2b. If f is continuous on $[a,b]$ and if m and M are numbers such that

$$m \leq f(t) \leq M, \quad \text{for} \quad a \leq t \leq b$$

then

$$m(b - a) \leq \int_a^{\bar{b}} f \leq M(b - a).$$

Proof. Consider the partition of $[a,b]$

$$\sigma_1 = [a,b]$$

and the corresponding upper and lower sums

$$U_1 = M(b - a); \quad L_1 = m(b - a).$$

The number $\int_a^{\overline{b}} f$ is the greatest lower bound of all upper sums of all partitions σ of $[a,b]$ so, since σ_1 is one such partition, we must have

$$\int_a^{\overline{b}} f \le U_1 = M(b - a).$$

The same argument also gives:

$$\int_{\underline{a}}^{b} f \ge m(b - a).$$

Recall that (see (2)):

$$\int_{\underline{a}}^{b} f \le \int_a^{\overline{b}} f$$

so that

$$m(b - a) \le \int_{\underline{a}}^{b} f \le \int_a^{\overline{b}} f \le M(b - a)$$

which gives the desired result.

The observant student will note that the continuity of f played no particular role in these lemmas, except to insure that f is bounded so that the upper and lower sums can be defined. Hence, both lemmas hold for an arbitrary bounded function f. In our third lemma, the continuity of f is essential.

Lemma A8-2c. Suppose f is continuous on $[a,b]$ and that x is a point of $[a,b]$. If ϵ is a given positive number then there is a positive number δ such that

$$\alpha - \beta \le \epsilon$$

where α and β are the respective maximum and minimum values of f on the closed interval

$$[a,b] \cap [x - \delta, x + \delta].$$

Proof. The fact that $[a,b] \cap [x - \delta, x + \delta]$ is a closed interval of positive length is easy to establish. (See Exercises A8-2, No. 1.) The lemma asserts that the difference between the maximum and minimum values of f on the interval $[x - \delta, x + \delta] \cap [a,b]$ can be made as small as we please, by choosing δ small enough. Its proof makes use of the definition of limit and Theorem 8-2b. Since f is continuous at x, we know that:

$$\lim_{t \to x} f(t) = f(x).$$

Therefore, if ϵ_1 is any given positive number, we can find a positive number δ_1 such that

$$|f(t) - f(x)| < \epsilon_1 \tag{7}$$

if t is in the domain of f and

$$0 < |t - x| < \delta_1. \tag{8}$$

The inequality (7) also holds if $x = t$, so (8) can be replaced by $0 \leq |t - x| < \delta_1$.

If ϵ is a given positive number, let $\epsilon_1 = \frac{\epsilon}{2}$. Choose $\delta_1 > 0$ so that

$$|f(t) - f(x)| < \epsilon_1 \tag{9}$$

if t is in the domain of f and

$$0 \leq |t - x| < \delta_1.$$

Let δ be a positive number smaller than δ_1. Thus, if $a \leq t \leq b$ and $x - \delta \leq t \leq x + \delta$ then t is in the domain of f and

$$0 \leq |t - x| \leq \delta < \delta_1$$

so that (9) holds. Let α and β be the maximum and minimum values of f on the interval $[a,b]$ $[x - \delta, x + \delta]$ and choose points c and d in this interval such that

$$\alpha = f(c); \quad \beta = f(d).$$

(The existence of c, d, α and β is guaranteed by Theorem 8-2b). Therefore,

$$|f(c) - f(x)| < \epsilon_1 \text{ and } |f(d) - f(x)| < \epsilon_1$$

so that

$$\begin{aligned}\alpha - \beta &= f(c) - f(d)\\ &= f(c) - f(x) - (f(d) - f(x)).\end{aligned}$$

The triangle inequality ((2) of Section A6-1) gives

$$\begin{aligned}\alpha - \beta &\leq |f(c) - f(x)| + |f(d) - f(x)|\\ &< \epsilon_1 + \epsilon_1 = \epsilon.\end{aligned}$$

This proves Lemma A8-2c.

Proof of Theorem A8-2. The function $\overline{F}$ is defined by

$$\overline{F}(x) = \int_a^{\overline{x}} f \quad \text{for } x \text{ in } [a,b].$$

Our purpose is to show that $\overline{F}'(x) = f(x)$, that is,

(10) $$\lim_{x' \to x} \frac{\overline{F}(x') - \overline{F}(x)}{x' - x} = f(x); \quad \text{for } x \text{ in } [a,b].$$

If ϵ is a given positive number, use Lemma A8-2c to find $\delta > 0$ so that

(11) $$\alpha - \beta < \epsilon$$

where α and β are the respective maximum and minimum values of f on the interval

(12) $$[a,b] \cap [x - \delta, x + \delta].$$

Suppose x' is in the domain of $\overline{F}$ and

(13) $$0 < |x' - x| < \delta$$

so that x' is in $[a,b]$ (since $\overline{F}$ is only defined on $[a,b]$) and hence x' is a point of (12). In particular, if $x' > x$, then $[x,x']$ is a subinterval of (12) (see Exercises A8-2, No. 2) and so

$$\beta \leq f(t) \leq \alpha \quad \text{if } t \in [x,x'].$$

Lemma A8-2b, then gives

(14) $$\beta(x' - x) \leq \int_x^{\overline{x}'} f \leq \alpha(x' - x).$$

In this case Lemma A8-2a gives

$$\int_a^{\overline{x}'} f = \int_a^{\overline{x}} + \int_x^{\overline{x}'} f$$

so that

$$\overline{F}(x') - \overline{F}(x) = \int_a^{\overline{x'}} f - \int_a^{\overline{x}} f = \int_x^{\overline{x'}} f$$

and hence (14) gives

$$\beta(x' - x) \leq \overline{F}(x') - \overline{F}(x) \leq \alpha(x' - x)$$

that is (since we are here assuming that $x' > x$)

$$\beta \leq \frac{\overline{F}(x') - \overline{F}(x)}{x' - x} \leq \alpha.$$

Subtract $f(x)$ throughout to obtain

$$\beta - f(x) \leq \frac{\overline{F}(x') - \overline{F}(x)}{x' - x} - f(x) \leq \alpha - f(x)$$

and now use the fact that

$$-f(x) \leq -\alpha \quad \text{and} \quad -f(x) \geq -\beta$$

(since x is in (12)) to obtain

$$\beta - \alpha \leq \frac{\overline{F}(x') - \overline{F}(x)}{x' - x} - f(x) \leq \alpha - \beta.$$

Using (11) we conclude that if x' is in the domain of $\overline{F}$ and (13) holds and if $x' > x$ then

$$\left|\frac{\overline{F}(x') - \overline{F}(x)}{x' - x} - f(x)\right| < \epsilon.$$

A similar result holds if $x' < x$ and we conclude that indeed (10) is true. This completes the proof of Theorem A8-2 and establishes that the integral of continuous function on a closed interval exists. The integral $\int_a^b f$ is then defined to be the common value of $\int_a^{\overline{b}} f$ and $\int_{\underline{a}}^b f$.

Exercises A8-2

1. Show that if $x \in [a,b]$ and $\delta > 0$ then $[a,b] \cap [x - \delta, x + \delta]$ is a closed interval. (Hint: Let a_1 be the larger of a and $x - \delta$, b_1, the smaller of b and $x + \delta$ and show that $[a_1,b_1] = [a,b] \cap [x - \delta, x + \delta]$).

2. Show that if $x' > x$ and $x' \in [a,b]$, $x \in [a,b]$ then $[x,x']$ is a subinterval of $[a,b]$.

3. Show that
$$\int_{\underline{a}}^{b} f = -\int_{a}^{\overline{b}} (-f).$$

4. Deduce from Number 3 and Theorem A8-2 that $\underline{F}' = f$ if f is continuous on $[a,b]$.

5. Show that if f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that
$$\int_{a}^{b} f = (b - a)f(c).$$
(Hint: Choose c_1 and d_1 in $[a,b]$ such that $f(c_1)$ and $f(d_1)$ are the respective maximum and minimum of f on $[a,b]$. Show that
$$f(d_1) \leq \frac{\int_a^b f}{b - a} \leq f(c_1)$$
and apply the Intermediate Value Theorem).

6. Use the Mean Value Theorem to show that Number 5 is true. Can you then choose c so that $a < c < b$?

7. Show that if f is continuous and nonnegative on $[a,b]$ with $a < b$ and if $f(x) > 0$ for some x in $[a,b]$ then $\int_a^b f > 0$.
(Hint: Show that there is a $\delta > 0$ and $m > 0$ such that $f(x) \geq m$ on $[a,b] \cap [x - \delta, x + \delta]$.

8. Deduce from Number 7 that if $f'(x) > 0$ for $a < x < b$ and f' is continuous on $[a,b]$ then f is strictly increasing on $[a,b]$.

9. Suppose

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2, & 1 < x \le 2 \end{cases}$$

(a) Show directly from the definition and properties of upper integrals that:

$$\overline{F}(x) = \overline{\int_0^x} f = \begin{cases} x, & 0 \le x \le 1 \\ 2x - 1, & 1 < x \le 2 \end{cases}$$

(b) Does $\overline{F}$ have a derivative at $x = 1$? Why doesn't this contradict Theorem A8-2?

10. Suppose f is bounded on $[a,b]$ and $\overline{F}(x) = \overline{\int_a^x} f$. Show that $\overline{F}$ is continuous on $[a,b]$. (Hint: Make use of Lemmas A8-2a, b, which hold for bounded functions.

Appendix 9

LOGARITHM AND EXPONENTIAL FUNCTIONS AS SOLUTIONS TO DIFFERENTIAL EQUATIONS

A9-1. The Logarithm as Integral

The logarithm function $\log_e$ is the unique solution to the problem

$$f'(x) = \frac{1}{x}, \quad f(1) = 0. \tag{1}$$

and can be expressed in the integral form

$$f(x) = \log_e x = \int_1^x \frac{1}{t}\,dt, \quad x > 0. \tag{2}$$

Our purpose in this section is to show how the properties of the logarithm function can be obtained by using the fact that it is the unique solution to (1) and that it is the area from 1 to x under the graph of $t \to \frac{1}{t}$.

In order not to be prejudiced by the known properties of the logarithm let us use the letter L to denote the function defined by

$$L(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0. \tag{3}$$

It will be shown that L has all the properties of the logarithm and that it is reasonable to write $L(x) = \log_e x$.

Certain elementary properties of L are easy to obtain from (3). First, note that

$$L(1) = 0, \tag{4}$$

since $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$. Second, the Area Theorem (Section 7-2) gives

$$L'(x) = \frac{1}{x}, \quad x > 0. \tag{5}$$

From (5) and the fact that $\frac{1}{x} > 0$ if $x > 0$ we conclude that $L'(x) > 0$ and

(6) L is a strictly increasing function.

In particular, since $L(1) = 0$, the values $L(x)$ for $0 < x < 1$ must be less than 0, while the values $L(x)$ for $x > 1$ must be greater than 0. That is,

$$L(x) < 0, \quad \text{if } 0 < x < 1 \tag{7}$$

and

(8) $$L(x) > 0, \text{ if } x > 1.$$

The second derivative of L is the derivative of $x \to \frac{1}{x}$ so that

(9) $$L''(x) = -\frac{1}{x^2}, \; x > 0.$$

The expression $-\frac{1}{x^2}$ is negative if $x \neq 0$, so that L is a concave function. Already we have enough information to know that the graph of L looks something like that shown in Figure A9-1a.

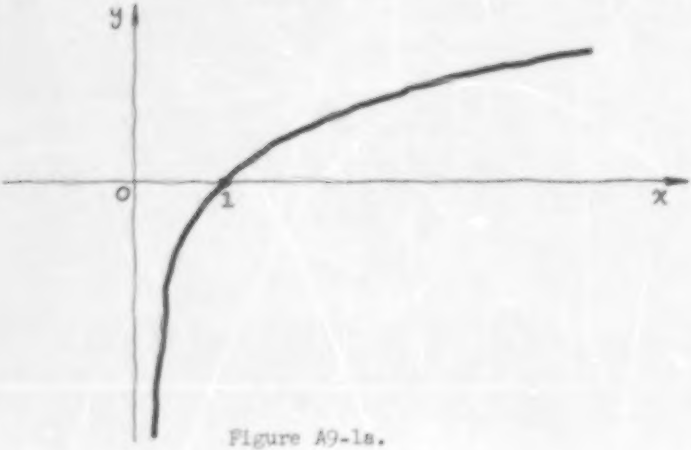

Figure A9-1a.

The graph of a strictly increasing, concave function, defined for $x > 0$, and passing through $(1,0)$.

The basic logarithm property

(10) $$L(ab) = L(a) + L(b), \; a > 0, \; b > 0$$

can be obtained by using the fact that $L = f$ is the unique solution to the problem

(11) $$f'(x) = \frac{1}{x}, \; f(1) = 0.$$

For suppose $a > 0$ and g is the function defined by

$$g(x) = L(ax) - L(a).$$

Certainly $g(1) = 0$. Furthermore, since $L(a)$ is a constant

$$Dg(x) = D(L(ax)).$$

The chain rule (with $u = ax$, $u' = a$), then gives

$$DL(ax) = \frac{1}{ax} \cdot a = \frac{1}{x}.$$

In other words, g is also a solution to problem (11). Since L is the only solution to this we must have $L = g$, that is

$$L(x) = L(ax) - L(a).$$

Adding $L(a)$ to both sides and replacing x by b then gives the result (10).

The formula $L(ab) = L(a) + L(b)$ tells us that the area under $t \rightarrow \frac{1}{t}$ from 1 to ab is the sum of the area from 1 to a and the area from 1 to b. (See Figure A9-1b.)

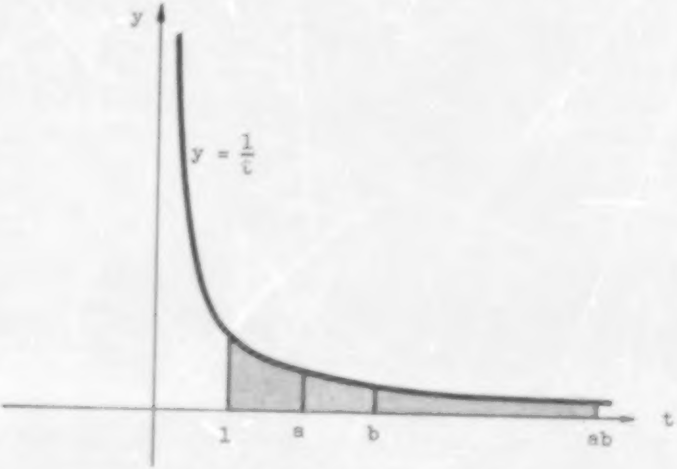

Figure A9-1b

The area of the shaded region is the area from 1 to a plus the area from 1 to b.

From (10) we have

$$L(a^2) = L(aa) = L(a) + L(a) = 2L(a)$$
$$L(a^3) = L(a^2a) = L(a^2) + L(a) = 2L(a) + L(a) = 3L(a)$$

and in general

$$L(a^n) = nL(a) \text{ if } n \text{ is any positive integer.}$$

Furthermore, if n is a positive integer and

$$b = a^{1/n}$$

then $b^n = a$, so that

$$L(a) = L(b^n) = nL(b) = nL(a^{1/n})$$

that is

$$L(a^{1/n}) = \frac{1}{n} L(a) \quad \text{if } n \text{ is a positive integer.}$$

Suppose r is a positive rational number so that $r = \frac{m}{n}$ where m and n are positive integers. Then

$$\begin{aligned} L(a^r) = L(a^{m/n}) &= L((a^{1/n})^m) \\ &= mL(a^{1/n}) \\ &= \frac{m}{n} L(a) \end{aligned}$$

that is,

$$L(a^r) = rL(a) \quad \text{if } r \text{ a positive rational number.}$$

This result will, in fact, be true for any rational number r. If $r = 0$, then

$$L(a^r) = L(a^0) = L(1) = 0 = 0L(a) = rL(a).$$

If $r < 0$, then $p = -r$ is positive and

$$a^r a^p = 1,$$

so that

$$\begin{aligned} 0 = L(1) &= L(a^r a^p) \\ &= L(a^r) + L(a^p) = L(a^r) + pL(a); \end{aligned}$$

that is,

$$L(a^r) = -pL(a) = rL(a).$$

In summary:

(12) $$L(a^r) = r\,L(a), \quad \text{if } a > 0 \text{ and } r \text{ is rational.}$$

It will now be shown that the range of L consists of all real numbers, and that L has an inverse function. In other words:

(13) $$\boxed{\begin{array}{l}\text{If } c \text{ is any real number, there is a unique} \\ \text{positive real number } d \text{ such that } L(d) = c.\end{array}}$$

To prove this we first show that for a given c there are positive numbers d_1 and d_2 such that

$$L(d_1) < c < L(d_2). \tag{14}$$

To do this, note that $L(2) > 0$ (from (8)). Hence, there is a negative integer n_1 and a positive integer n_2 such that

$$n_1\, L(2) < c < n_2\, L(2)$$

We can then choose

$$d_1 = 2^{n_1} \text{ and } d_2 = 2^{n_2}.$$

It follows that

$$L(d_1) = L(2^{n_1}) = n_1\, L(2) < c$$

and

$$L(d_2) = L(2^{n_2}) = n_2\, L(2) > c$$

so that d_1 and d_2 are positive numbers which satisfy (14). The function L is differentiable for each $x > 0$; therefore, it is continuous for each $x > 0$ (Section 8-1). The Intermediate Value Theorem (Section 8-2) implies that there is a positive real number d between d_1 and d_2 such that

$$L(d) = c.$$

Furthermore, d must be unique since L is strictly increasing. This completes the proof of (13).

Exercises A9-1

1. Use upper and lower sums to show that

$$\frac{1}{3} + \frac{1}{4} < L(2) < \frac{1}{2} + \frac{1}{3} .$$

2. (a) Show that for each integer $n > 1$

$$\frac{1}{n} + L(n) < 1 + \frac{1}{2} + \ldots + \frac{1}{n} < 1 + L(n).$$

(Hint: Use upper and lower sums to estimate $\int_1^n \frac{1}{t}\,dt$.)

(b) Estimate $\displaystyle\sum_{n=1}^{10^{100}} \frac{1}{n}$.

3. (a) Show that if $a > 1$

$$1 - \frac{1}{a} < L(a) < a - 1.$$

(b) Show that if $a > 1$

$$L(2a) > L(a) + \frac{1}{2} .$$

(c) Show that if $a > 1$ then

$$L(a) < 2\sqrt{a}.$$

(Hint: $L(a) = 2L(\sqrt{a})$.)

4. Show that $\lim\limits_{x \to \infty} \frac{L(x)}{x} = 0$. (Hint: Use No. 3(c).)

5. Find $f'(x)$ for each of the following

(a) $f(x) = L\left(\sqrt{\frac{x - 1}{x + 1}}\right)$

(b) $f(x) = L(x\sqrt{1 - x})$

(c) $f(x) = L(L(x))$

6. Sketch the graph of $x \to x\,L(x)$, using its derivative.

A9-2. The Exponential Functions

Denoting the inverse of L by E, it follows that E is defined for each real number c by

$$E(c) = d \text{ if } L(d) = c,$$

from which we have

$$(15) \qquad \boxed{\begin{array}{l} E(L(d)) = d \text{ for each } d > 0 \\ L(E(c)) = c \text{ for each } c \end{array}}$$

The values $E(x)$ of the function E are positive (because it is the inverse of a function whose domain consists of positive numbers). Furthermore, E is strictly increasing and continuous because it is the inverse of a strictly increasing continuous function. (See (3)ff, Section 8-11.) For the function E the two results (10) and (12) now take the form:

$$(16) \qquad \boxed{\begin{array}{l} E(a + b) = E(a)\,E(b) \\ E(a^r) = E(a)^r \text{ for any rational number } r \end{array}}$$

For example, to show that $E(a + b) = E(a)E(b)$, we note that

$$L(E(a + b)) = a + b$$

and that

$$\begin{aligned} L(E(a)E(b)) &= L(E(a)) + L(E(b)) \\ &= a + b \end{aligned}$$

so that $L(E(a + b)) = L(E(a)E(b))$. Since L is strictly increasing we must have $E(a + b) = E(a)E(b)$.

If r is a rational number then (15) tells us that

$$a^r = E(L(a^r)).$$

Since $L(a^r) = r\,L(a)$ we therefore have:

$$(17) \qquad a^r = E(r\,L(a)), \text{ if } r \text{ is rational and } a > 0.$$

Let us now define a^x for $a > 0$ and x arbitrary by

$$(18) \qquad \boxed{a^x = E(x\,L(a)),}$$

that is, by extending (17) to all real numbers x. We shall show that this definition agrees with the definition of a^x used in Chapters 5 and 6.

The laws of exponents hold for our new definition (18). For example,

$$a^{x+y} = E((x + y)L(a)) = E(xL(a) + yL(a))$$

so that (16) gives

$$a^{x+y} = E(xL(a))E(yL(a)) = a^x a^y.$$

We prove that $(a^x)^y = a^{xy}$ as follows. From (18) we have

$$(a^x)^y = E(yL(a^x)).$$

We replace a^x by $E(xL(a))$ to obtain

$$(a^x)^y = E(yL(E(xL(a))).$$

Now use the fact that

$$L(E(xL(a))) = x\ L(a)$$

(an application of the second formula of (15)) to obtain

$$(a^x)^y = E(yx\ L(a)) = E(xy\ L(a)).$$

Now use the definition of powers (18) again to write

$$E(xy\ L(a)) = a^{xy}.$$

We conclude that $(a^x)^y = a^{xy}$.

Note that if $a > 1$, the function $x \to a^x$ is strictly increasing. For if $a > 1$ and $x_1 < x_2$, then $L(a) > 0$ so that

$$x_1\ L(a) < x_2\ L(a).$$

Since E is strictly increasing, we must have

$$a^{x_1} = E(x_1\ L(a)) < E(x_2\ L(a)) = a^{x_2}.$$

A similar argument shows that $x \to a^x$ is strictly decreasing if $0 < a < 1$. The function $x \to a^x$ is continuous, for

$$\begin{aligned} \lim_{x \to b} a^x &= \lim_{x \to b} E(x\ L(a)) \\ &= E(\lim_{x \to b} x\ L(a)) \end{aligned}$$

(since E is continuous). Since $\lim\limits_{x \to b} x\,L(a) = b\,L(a)$ and $E(b\,L(a)) = a^b$ we indeed have

$$\lim_{x \to b} a^x = a^b.$$

In summary, if arbitrary powers are defined by (18) then the laws of exponents hold, the function $x \to a^x$ is continuous, and is strictly increasing if $a > 1$, strictly decreasing if $0 < a < 1$. It appears that, indeed, the definition (18)) results in desirable properties for exponential functions.

The results of Section 8-11 enable us to find the derivative of E, and hence, using the chain rule, the derivative of $x \to a^x$. Since the derivative of L is the function $x \to \frac{1}{x}$, $x > 0$ which has only positive values, we know that

$$E'(x) = \frac{1}{L'(E(x))}$$

(see (5), Section 8-11). The formula $L'\colon x \to \frac{1}{x}$ then gives

$$L'(E(x)) = \frac{1}{E(x)}$$

so that

$$E'(x) = \frac{1}{L'(E(x))} = E(x).$$

In summary, the function E is its own derivative. Therefore, $f(x) = E(x)$ is a solution to the problem

$$f' = f;\ f(0) = 1. \tag{19}$$

In our previous discussions (Chapters 5, 6) it was shown that if

$$\lim_{h \to 0} \frac{2^h - 1}{h} \text{ exists and is } k \tag{20}$$

and if e is defined to be $2^{1/k}$, and e^x to be $2^{(1/k)x}$, then

$$x \to e^x \text{ must be a solution to (19).}$$

We conclude that if (20) holds, then E and the function $x \to e^x$ must be the same function, that is

$$E(x) = e^x, \quad \text{for all } x. \tag{21}$$

In our setting, arbitrary powers are defined by (18). Let us show that indeed (20) and (21) are true if we use (18) to define powers.

First we use the result $E' = E$ and the chain rule to find the derivative of $f : x \to a^x$. We have

$$f(x) = a^x = E(x\,L(a)).$$

Put $u(x) = x\,L(a)$, $g = E$, so that

$$f(x) = g(u(x))$$

and, hence,

$$f'(x) = Da^x = g'(u(x))u'(x).$$

Since $g' = g = E$ and $u'(x) = L(a)$ we have

$$\begin{aligned} Da^x &= E(x\,L(a)) \cdot L(a) \\ &= a^x \cdot L(a) \end{aligned}$$

that is

$$f'(x) = a^x\,L(a), \quad \text{if } f : x \to a^x. \tag{22}$$

In particular,

$$f'(0) = a^0\,L(a) = L(a).$$

Expressing the derivative as the limit of a difference quotient, we have:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} \\ &= \lim_{h \to 0} \frac{a^h - 1}{h}. \end{aligned}$$

We conclude that $\lim\limits_{h \to 0} \frac{a^h - 1}{h}$ indeed exists and, in fact

$$L(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Put

$$\lim_{h \to 0} \frac{2^h - 1}{h} = L(2) \quad \text{and} \quad e = 2^{1/k}$$

that is, since we are using (18) to define powers,

$$e = E(\tfrac{1}{k} L(2)).$$

Since $L(2) = k$, this means that

$$e = E(1), \quad \text{so that} \quad L(e) = 1.$$

Thus, indeed we have

$$e^x = E(x\, L(e)) = E(x).$$

Exercises A9-2

1. Use the definition (18) to find $f'(x)$ where

   (a) $f(x) = (1 - x)^x$

   (b) $f(x) = (L(x))^x$

   (c) $f(x) = x^{1/x}$

2. Find the minimum value of $x \to x^x$.

3. Show that if $y' = cy$ where c is a constant then there is a constant K such that

$$y = KE(cx).$$

   (Hint: Put $z = E(-cx)y$ and show that $z' = 0$.)

4. Recall that if x_1 is an initial estimate to a zero r of f then Newton's Method (Section 2-10), under suitable conditions gives the better estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and the subsequent estimates

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

This can be used to estimate e, the zero of $f(x) = L(x) - 1$. Using $x_1 = 2$ and $L(2) \approx 0.7$ find x_2 and x_3.

5. (a) Show that

$$e = \lim_{h \to 0} (1 + h)^{1/h}$$

$$\text{(Hint: } L'(1) = 1 = \lim_{h \to 0} \frac{L(1 + h) - L(1)}{h}$$

$$= \lim_{h \to 0} L((1 + h)^{1/h})$$

(b) Show that $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$

(c) Show that $e^a = \lim_{n \to \infty} (1 + \frac{a}{n})^n$

The Circular Functions

The sine and cosine functions can be constructed as inverses of solutions to

$$y' = \frac{1}{\sqrt{1 - x^2}} .$$

The following exercises outline this construction. As in our logarithm discussion we shall introduce new symbols for these functions, then show that they are the desired functions. We let A be the function defined for $|x| < 1$ by

$$A(x) = \int_0^x \frac{1}{\sqrt{1 - t^2}} \, dt.$$

6. Find A' and show that A is strictly increasing and continuous, and, hence, has an inverse S.

7. (a) What is $S(0)$?

   (b) Show that

$$S' = \sqrt{1 - S^2}$$

   by using the formula for the derivative of the inverse.

   (c) What is $S'(0)$?

   (d) Show that $S'' + S = 0$.

8. Let $C = S'$ and use Number 7.

   (a) Show that $C'' + C = 0$.

   (b) Show that $C' = -S$.

   (c) What is $C(0)$? $C'(0)$?

   (d) Show that $[C(x)]^2 + [S(x)]^2 = 1$.

9. Show that if $y'' + y = 0$, $y(0) = 0$ and $y'(0) = 1$ then $y = S(x)$. (Hint: Put $z = y - S(x)$ and use the fact that

$$0 = (z'' + z)z' = \frac{1}{2} D((z')^2 + z^2)$$

   and $z(0) = z'(0) = 0$ to show that $z = 0$.)

10. Use Number 9 to show that

$$S(x + a) = S(x)C(a) + S(a)C(x)$$

   if x, a, and $x + a$ are in the domain of S.

Remark. The above defines the functions S and C only for x near zero (that is, for x in the range of A where A is defined on the interval $-1 < t < 1$). The intuitive discussion of Chapters 3 and 4 showed that $y = \sin x$ is a solution to $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$ so we are able to conclude that $S(x) = \sin x$ for x near zero. A method for extending the functions S and C to all x is discussed in Appendix 8, SMSG Calculus, Volume 2.